4.5

Solving Systems with Matrices

Objectives:

1. **Solve a system of linear equations using matrices.**

Writing the Augmented Matrix of a System of Equations

A **matrix** is an array of numbers can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an **augmented matrix**.

For example, consider the following 2 × 2 system of equations.

$$\begin{aligned} 3x + 4y &= 7 \\ 4x - 2y &= 5 \end{aligned}$$

We can write this system as an augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 4 & 7 \\ 4 & -2 & 5 \end{array}\right]$$

A three-by-three **system of equations** such as

$$\begin{aligned} 3x - y - z &= 0 \\ x + y &= 5 \\ 2x - 3z &= 2 \end{aligned}$$

can be represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2 \end{array}\right]$$

Notice that the matrix is written so that the variables line up in their own columns: x-terms go in the first column, y-terms in the second column, and z-terms in the third column. It is very important that each equation is written in standard form $ax + by + cz = d$ so that the variables line up. When there is a missing variable term in an equation, the coefficient is 0.

The calculator can be used to solve systems of linear equations by performing operations on the rows of augmented matrix to change the matrix to **row reduced echelon form**. The following matrices are in row reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Notice that the coefficients are all zero with a diagonal of ones. This form of the matrix makes it easy to see the solution to the system. Translating each row of the matrix into the equation that it represents shows us that the solution to the system is easy to identify.

Row Reduced Echelon Form		**Equivalent Equations**		**Simplified Equations**		**Solution to the System**
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \end{bmatrix}$	$\rightarrow$	$\begin{matrix} 1x + 0y = 0 \\ 0x + 1y = 6 \end{matrix}$	$\rightarrow$	$\begin{matrix} x + 0 = 0 \\ 0 + y = 6 \end{matrix}$	$\rightarrow$	$\begin{matrix} x = 0 \\ y = 6 \end{matrix}$
$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$	$\rightarrow$	$\begin{matrix} 1x + 0y + 0z = 3 \\ 0x + 1y + 0z = -2 \\ 0x + 0y + 1z = 7 \end{matrix}$	$\rightarrow$	$\begin{matrix} x + 0 + 0 = 3 \\ 0 + y + 0 = -2 \\ 0 + 0 + z = 7 \end{matrix}$	$\rightarrow$	$\begin{matrix} x = 3 \\ y = -2 \\ z = 7 \end{matrix}$

Notice how the last column in the row reduced echelon form corresponds to the solution to the system.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \end{bmatrix} \rightarrow \begin{matrix} x = 0 \\ y = 6 \end{matrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{bmatrix} \rightarrow \begin{matrix} x = 3 \\ y = -2 \\ z = 7 \end{matrix}$$

The operations for changing an augmented matrix to its row reduced echelon form are outside the scope of this class. However, the calculator is capable of performing these operations.

There are four steps to reducing a matrix to row reduced echelon form.

1. Change both equation to standard form $Ax + By = C$,
2. Represent the system with an augmented matrix,
3. Enter the matrix into the calculator,
4. Convert the matrix to row reduced echelon form.

Example 1: Solve the following system of equations.

$$y = -3x + 14$$
$$2x - y = 6$$

Answer:

Step 1: Change both equations to standard form.

First, represent the system with an augmented matrix. To do this, change each equation in the system to standard form Ax + By = C. The first equation $y = -3x + 14$ is not in standard form, so add 3x to both sides of the equation.

$$\begin{array}{rcl} & y = & -3x + 14 \\ +3x & & +3x \\ 3x & + y = & 14 \end{array}$$

The second equation $2x - y = 6$ is already in standard form. The two equations in standard form are:

$$3x + y = 14$$
$$2x - y = 6$$

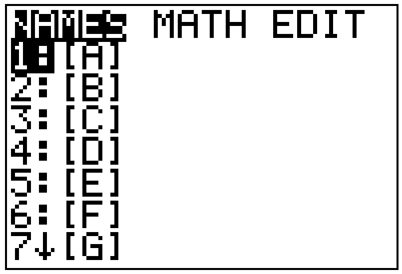
Step 2: Represent the system with an augmented matrix.

Now represent this system with an augmented matrix. Remember to place all the x coefficients in the first column and the y coefficients in the second column.

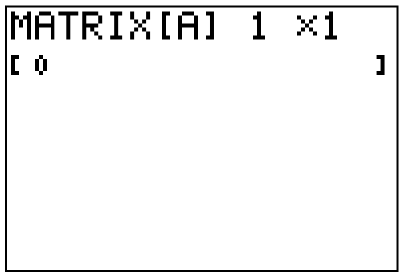
$$\begin{array}{l} 3x + y = 14 \\ 2x - y = 6 \end{array} \quad \rightarrow \quad \begin{bmatrix} 3 & 1 & 14 \\ 2 & -1 & 6 \end{bmatrix}$$

Step 3: Enter the matrix into the calculator.

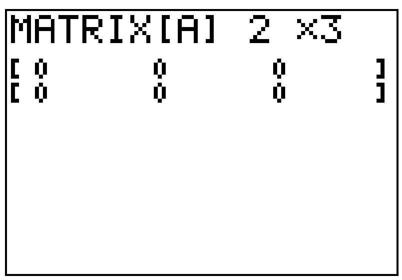
To enter the matrix into the calculator, either press the **MATRX** button, or if the calculator does not have a **MATRX** button press the **2ND key** and then the $\boldsymbol{x^{-1}}$ key to enter the **MATRIX** menu. The calculator screen should look similar to the image below.



Use the **right arrow** key to cursor over to **EDIT** with the **right arrow** key. Press **1** to edit matrix [A]. (Note: the letter A is just a label for a matrix. There is space enough for ten matrices on your calculator, labeled A through J).



Enter the size of the matrix. The matrix for this system has 2 rows and 3 columns, so press **2** and **ENTER**, then **3** and **ENTER**.



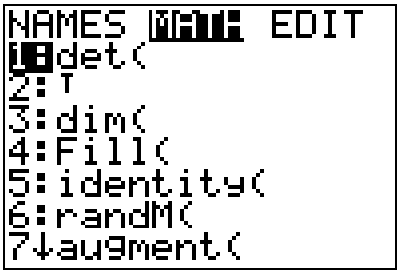
Enter the coefficients and constants in the augmented matrix into the calculator. After entering a number, press **ENTER** to move to the next entry. You can also use the arrow keys to navigate from entry to entry.

$$\begin{aligned} 3x + y &= 14 \\ 2x - y &= 6 \end{aligned} \rightarrow \begin{bmatrix} 3 & 1 & 14 \\ 2 & -1 & 6 \end{bmatrix} \rightarrow$$

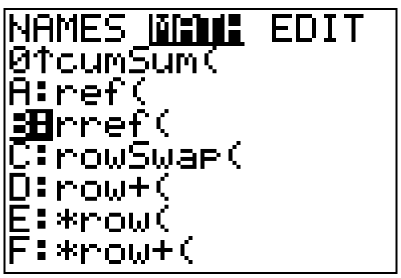
Step 4: Convert the matrix to row reduced echelon form.

Now that we have entered the matrix into the calculator, we want to reduce it to row reduced echelon form. Press **2ND** and **MODE** to **QUIT**.

Next press **2ND** and $\boldsymbol{x^{-1}}$ (or press the **MATRX** button) to enter the **MATRIX** menu. Then curser right to the **MATH** menu.



Use the arrow keys to highlight **rref** (short for row reduced echelon form) and press **ENTER**.



The last step is to input the name of the matrix we want to row reduce into the calculator. Press **2ND** and $\boldsymbol{x^{-1}}$ (or press the MATRX button) to enter the **MATRIX** menu. Then press **1** to select matrix A. Close the parentheses and press **ENTER.**

The first row of the matrix represents the equation

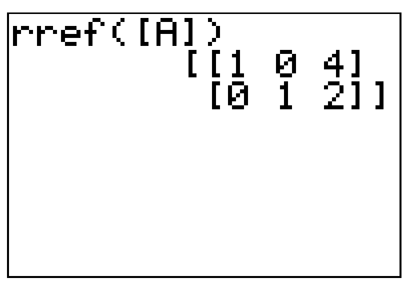
$$1x + 0y = 4.$$

Which can be simplified to

$$x + 0 = 4$$

or

$$\mathbf{x = 4.}$$



Similarly, the second row of the matrix represents the equation

$$0x + 1y = 2 \quad \text{or} \quad \mathbf{y = 2.}$$

The solutions to the system is $x = 4$ and $y = 2$ or **(4, 2).**

Example 2: Solve the following system of equations.

$$4x + 5y = 31$$
$$x = 10y - 26$$

Answer:

Change both equations to standard form Ax + By = C.

$$4x + 5y = 31$$
$$x - 10y = -26$$

Write the corresponding augmented matrix.

$$\begin{bmatrix} 4 & 5 & 31 \\ 1 & -10 & -26 \end{bmatrix}$$

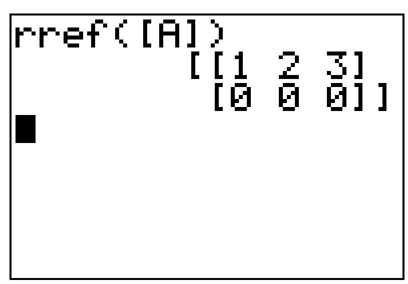
Use the steps outlined in **Example 1** to find the row reduced echelon form of the matrix.

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

The solutions to the system is $x = 4$ and $y = 3$ or **(4, 3).**

Dependent and Inconsistent Systems

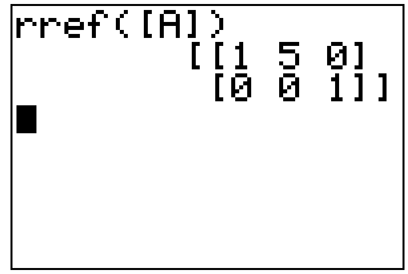
If a system was **dependent**, the calculator would display a row of all 0's, like the image to the right.



The last row translates to $0x + 0y = 0$, which is always true and the solution is that there are **infinitely many solutions.**

If the system was **inconsistent**, the calculator would display a row of 0's with a 1 in the right column, like in this image.

That last row represents the equation 0x + 0y = 1, which is impossible and the system has **no solution**.



4.4 Practice – Solving Systems with Matrices

For the following exercises, write the augmented matrix for the linear system.

6. $\begin{aligned} 8x - 37y &= 8 \\ 2x + 12y &= 3 \end{aligned}$

7. $\begin{aligned} 16y &= 4 \\ 9x - y &= 2 \end{aligned}$

8. $\begin{aligned} 3x + 2y + 10z &= 3 \\ -6x + 2y + 5z &= 13 \\ 4x + z &= 18 \end{aligned}$

9. $\begin{aligned} x + 5y + 8z &= 19 \\ 12x + 3y &= 4 \\ 3x + 4y + 9z &= -7 \end{aligned}$

10. $\begin{aligned} 6x + 12y + 16z &= 4 \\ 19x - 5y + 3z &= -9 \\ x + 2y &= -8 \end{aligned}$

For the following exercises, write the linear system from the augmented matrix.

11. $\left[\begin{array}{cc|c} -2 & 5 & 5 \\ 6 & -18 & 26 \end{array}\right]$

12. $\left[\begin{array}{cc|c} 3 & 4 & 10 \\ 10 & 17 & 439 \end{array}\right]$

13. $\left[\begin{array}{ccc|c} 3 & 2 & 0 & 3 \\ -1 & -9 & 4 & -1 \\ 8 & 5 & 7 & 8 \end{array}\right]$

14. $\left[\begin{array}{ccc|c} 8 & 29 & 1 & 43 \\ -1 & 7 & 5 & 38 \\ 0 & 0 & 3 & 10 \end{array}\right]$

15. $\left[\begin{array}{ccc|c} 4 & 5 & -2 & 12 \\ 0 & 1 & 58 & 2 \\ 8 & 7 & -3 & -5 \end{array}\right]$

For the following exercises, solve the system by Gaussian elimination.

16. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right]$

17. $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 0 & 2 \end{array}\right]$

18. $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$

19. $\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 4 & -5 & 6 \end{array}\right]$

20. $\left[\begin{array}{cc|c} -2 & 0 & 1 \\ 0 & 2 & -1 \end{array}\right]$

21. $\begin{aligned} 2x - 3y &= -9 \\ 5x + 4y &= 58 \end{aligned}$

22. $\begin{aligned} 6x + 2y &= -4 \\ 3x + 4y &= -17 \end{aligned}$

23. $\begin{aligned} 2x + 3y &= 12 \\ 4x + y &= 14 \end{aligned}$

24. $\begin{aligned} -4x - 3y &= -2 \\ 3x - 5y &= -13 \end{aligned}$

25. $\begin{aligned} -5x + 8y &= 3 \\ 10x + 6y &= 5 \end{aligned}$

26. $\begin{aligned} 3x + 4y &= 12 \\ -6x - 8y &= -24 \end{aligned}$

27. $\begin{aligned} -60x + 45y &= 12 \\ 20x - 15y &= -4 \end{aligned}$

28. $\begin{aligned} 11x + 10y &= 43 \\ 15x + 20y &= 65 \end{aligned}$

29. $\begin{aligned} 2x - y &= 2 \\ 3x + 2y &= 17 \end{aligned}$

30. $\begin{aligned} -1.06x - 2.25y &= 5.51 \\ -5.03x - 1.08y &= 5.40 \end{aligned}$

31. $\begin{aligned} \frac{3}{4}x - \frac{3}{5}y &= 4 \\ \frac{1}{4}x + \frac{2}{3}y &= 1 \end{aligned}$

32. $\begin{aligned} \frac{1}{4}x - \frac{2}{3}y &= -1 \\ \frac{1}{2}x + \frac{1}{3}y &= 3 \end{aligned}$

33. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 31 \\ 0 & 1 & 1 & 45 \\ 0 & 0 & 1 & 87 \end{array}\right]$

34. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 50 \\ 1 & 1 & 0 & 20 \\ 0 & 1 & 1 & -90 \end{array}\right]$

35. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{array}\right]$

36. $\left[\begin{array}{ccc|c} -0.1 & 0.3 & -0.1 & 0.2 \\ -0.4 & 0.2 & 0.1 & 0.8 \\ 0.6 & 0.1 & 0.7 & -0.8 \end{array}\right]$

37. $$\begin{aligned} -2x + 3y - 2z &= 3 \\ 4x + 2y - z &= 9 \\ 4x - 8y + 2z &= -6 \end{aligned}$$

38. $$\begin{aligned} x + y - 4z &= -4 \\ 5x - 3y - 2z &= 0 \\ 2x + 6y + 7z &= 30 \end{aligned}$$

39. $$\begin{aligned} 2x + 3y + 2z &= 1 \\ -4x - 6y - 4z &= -2 \\ 10x + 15y + 10z &= 5 \end{aligned}$$

40. $$\begin{aligned} x + 2y - z &= 1 \\ -x - 2y + 2z &= -2 \\ 3x + 6y - 3z &= 5 \end{aligned}$$

41. $$\begin{aligned} x + 2y - z &= 1 \\ -x - 2y + 2z &= -2 \\ 3x + 6y - 3z &= 3 \end{aligned}$$

42. $$\begin{aligned} x + y &= 2 \\ x + z &= 1 \\ -y - z &= -3 \end{aligned}$$

43. $$\begin{aligned} x + y + z &= 100 \\ x + 2z &= 125 \\ -y + 2z &= 25 \end{aligned}$$

44. $$\begin{aligned} \frac{1}{4}x - \frac{2}{3}z &= -\frac{1}{2} \\ \frac{1}{5}x + \frac{1}{3}y &= \frac{4}{7} \\ \frac{1}{5}y - \frac{1}{3}z &= \frac{2}{9} \end{aligned}$$

45. $$\begin{aligned} -\frac{1}{2}x + \frac{1}{2}y + \frac{1}{7}z &= -\frac{53}{14} \\ \frac{1}{2}x - \frac{1}{2}y + \frac{1}{4}z &= 3 \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{3}z &= \frac{23}{15} \end{aligned}$$

46. $$\begin{aligned} -\frac{1}{2}x - \frac{1}{3}y + \frac{1}{4}z &= -\frac{29}{6} \\ \frac{1}{5}x + \frac{1}{6}y - \frac{1}{7}z &= \frac{431}{210} \\ -\frac{1}{8}x + \frac{1}{9}y + \frac{1}{10}z &= -\frac{49}{45} \end{aligned}$$

For the following exercises, use Gaussian elimination to solve the system.

47. $$\begin{aligned} \frac{x-1}{7} + \frac{y-2}{8} + \frac{z-3}{4} &= 0 \\ x + y + z &= 6 \\ \frac{x+2}{3} + 2y + \frac{z-3}{3} &= 5 \end{aligned}$$

48. $$\begin{aligned} \frac{x-1}{4} - \frac{y+1}{4} + 3z &= -1 \\ \frac{x+5}{2} + \frac{y+7}{4} - z &= 4 \\ x + y - \frac{z-2}{2} &= 1 \end{aligned}$$

49. $$\begin{aligned} \frac{x-3}{4} - \frac{y-1}{3} + 2z &= -1 \\ \frac{x+5}{2} + \frac{y+5}{2} + \frac{z+5}{2} &= 8 \\ x + y + z &= 1 \end{aligned}$$

50. $$\begin{aligned} \frac{x-3}{10} + \frac{y+3}{2} - 2z &= 3 \\ \frac{x+5}{4} - \frac{y-1}{8} + z &= \frac{3}{2} \\ \frac{x-1}{4} + \frac{y+4}{2} + 3z &= \frac{3}{2} \end{aligned}$$

51. $$\begin{aligned} \frac{x-3}{4} - \frac{y-1}{3} + 2z &= -1 \\ \frac{x+5}{2} + \frac{y+5}{2} + \frac{z+5}{2} &= 7 \\ x + y + z &= 1 \end{aligned}$$

Answers – Solving Systems with Matrices

7. $\left[\begin{array}{cc|c} 0 & 16 & 4 \\ 9 & -1 & 2 \end{array}\right]$ **9.** $\left[\begin{array}{ccc|c} 1 & 5 & 8 & 19 \\ 12 & 3 & 0 & 4 \\ 3 & 4 & 9 & -7 \end{array}\right]$

11. $\begin{aligned} -2x + 5y &= 5 \\ 6x - 18y &= 26 \end{aligned}$

13. $\begin{aligned} 3x + 2y &= 13 \\ -x - 9y + 4z &= 53 \\ 8x + 5y + 7z &= 80 \end{aligned}$

15. $\begin{aligned} 4x + 5y - 2z &= 12 \\ y + 58z &= 2 \\ 8x + 7y - 3z &= -5 \end{aligned}$

17. No solutions

19. $(-1, -2)$

21. $(6, 7)$

23. $(3, 2)$ **25.** $\left(\frac{1}{5}, \frac{1}{2}\right)$ **27.** $\left(x, \frac{4}{15}(5x + 1)\right)$ **29.** $(3, 4)$

31. $\left(\frac{196}{39}, -\frac{5}{13}\right)$ **33.** $(31, -42, 87)$ **35.** $\left(\frac{21}{40}, \frac{1}{20}, \frac{9}{8}\right)$

37. $\left(\frac{18}{13}, \frac{15}{13}, -\frac{15}{13}\right)$ **39.** $\left(x, y, \frac{1}{2} - x - \frac{3}{2}y\right)$

41. $\left(x, -\frac{x}{2}, -1\right)$ **43.** $(125, -25, 0)$ **45.** $(8, 1, -2)$

47. $(1, 2, 3)$ **49.** $\left(-4z + \frac{17}{7}, 3z - \frac{10}{7}, z\right)$

51. No solutions exist.