Beginning Algebra

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For Our Students

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Pre-Algebra - Sets of Numbers

Objectives: **1. Construct a number line and graph points on it.**
2. Use a number line to determine the order of real numbers.
3. Determine the opposite of a real number.
4. Determine the absolute value of a real number.

Definitions: A set is a collection of objects, typically grouped within braces { }, where each object is called an element. For example, {red, green, blue} is a set of colors. A subset is a set consisting of elements that belong to a given set. For example, {green, blue} is a subset of the color set above. A set with no elements is called the empty set and has its own special notation, { } or ∅. When studying mathematics, we focus on special sets of numbers. The set of natural (or counting) numbers, denoted **N**, is combined with zero.

$$\{0, 1, 2, 3, 4, 5, ...\} \quad \textit{Whole numbers}$$

The three periods (…) is called an ellipsis and indicates that the numbers continue without bound. The set of whole numbers, denoted *W*, is the set of natural numbers combined with zero.

The set of integers, denoted **Z**, consists of both positive and negative whole numbers, as well as zero.

$$\{..., -3, -2, -1, 0, 1, 2, 3, ...\} \quad \textit{Integers}$$

Notice that the sets of natural and whole numbers are both subsets of the set of integers.

Rational numbers, denoted **Q**, are defined as any number of the form *ab*, where *a* and *b* are integers and *b* is nonzero. Decimals that repeat or terminate are rational. For example,

$$0.7 = \frac{7}{10} \quad \text{and} \quad 0.\overline{3} = 0.3333\ldots = \frac{1}{3}$$

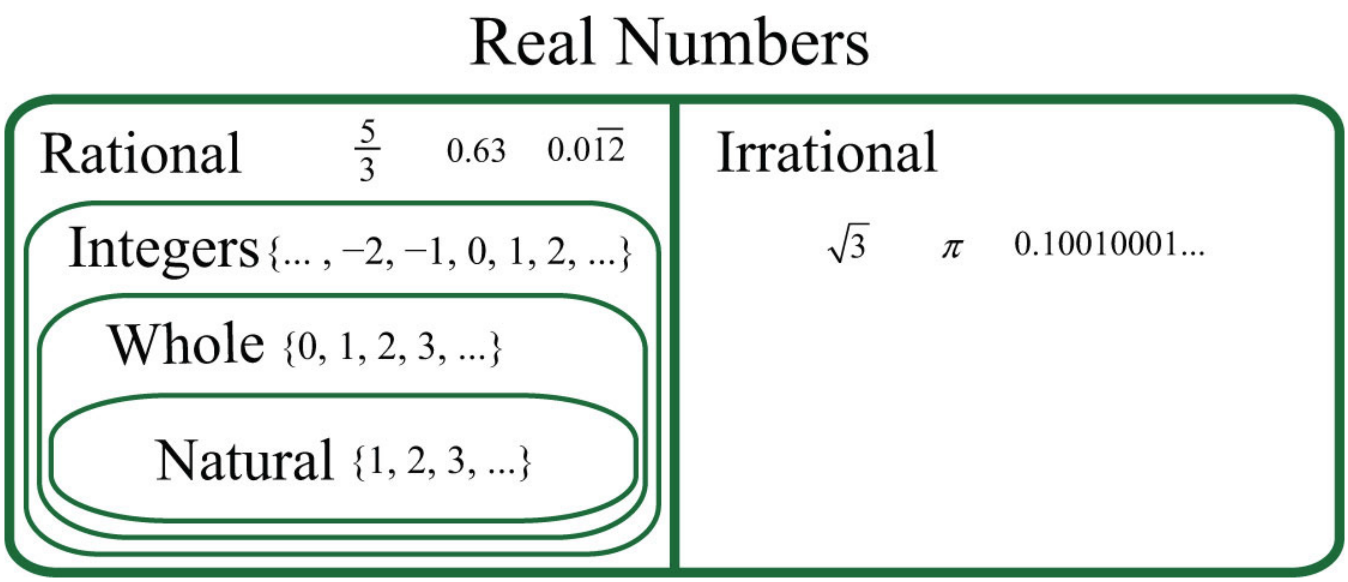
The set of integers is a subset of the set of rational numbers because every integer can be expressed as a ratio of the integer and 1. In other words, any integer can be written over 1 and can be considered a rational number. For example,

$$5 = \frac{5}{1}$$

Irrational numbers are defined as any number that cannot be written as a ratio of two integers. Nonterminating decimals that do not repeat are irrational. For example,

$$\pi = 3.14159\ldots \quad \text{and} \quad \sqrt{2} = 1.41421\ldots$$

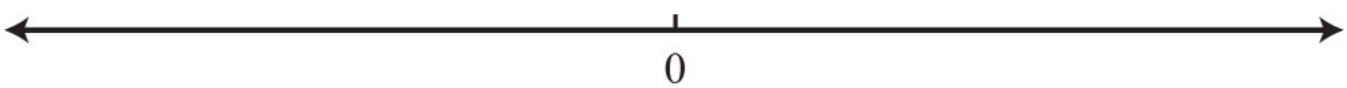
The set of real numbers, denoted **R**, is defined as the set of all rational numbers combined with the set of all irrational numbers. Therefore, all the numbers defined so far are subsets of the set of real numbers. In summary,



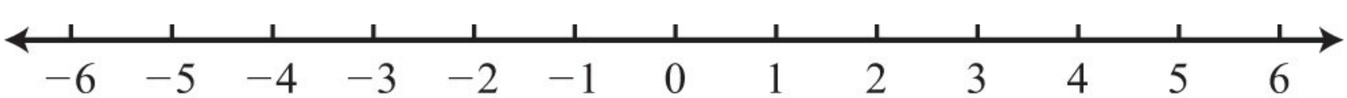
Number Line

A real number line, or simply number line, allows us to visually display real numbers by associating them with unique points on a line. The real number associated with a point is called a coordinate. A point on the real number line that is associated with a coordinate is called its graph.

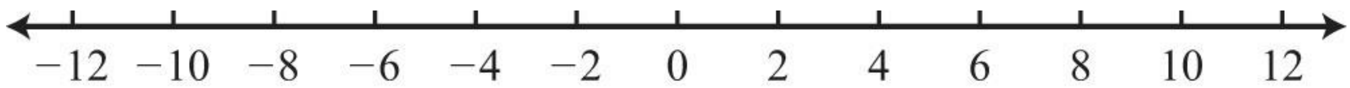
To construct a number line, draw a horizontal line with arrows on both ends to indicate that it continues without bound. Next, choose any point to represent the number zero; this point is called the origin.

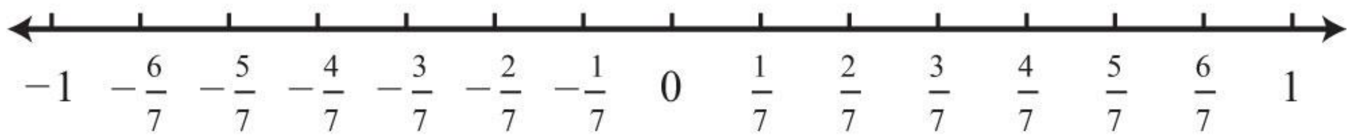


Mark off consistent lengths on both sides of the origin and label each tick mark to define the scale. Positive real numbers lie to the right of the origin and negative real numbers lie to the left. The number zero (0) is neither positive nor negative. Typically, each tick represents one unit.

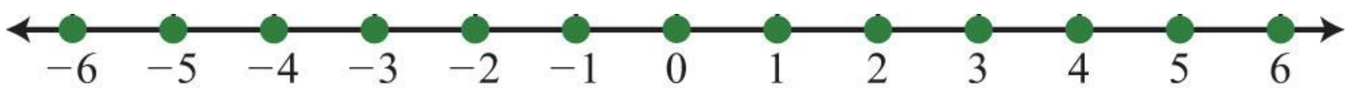


As illustrated below, the scale need not always be one unit. In the first number line, each tick mark represents two units. In the second, each tick mark represents a fraction, 1/7.



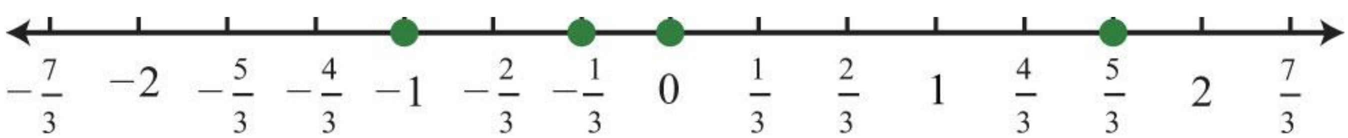


The graph of each real number is shown as a dot at the appropriate point on the number line. A partial graph of the set of integers **Z** follows:



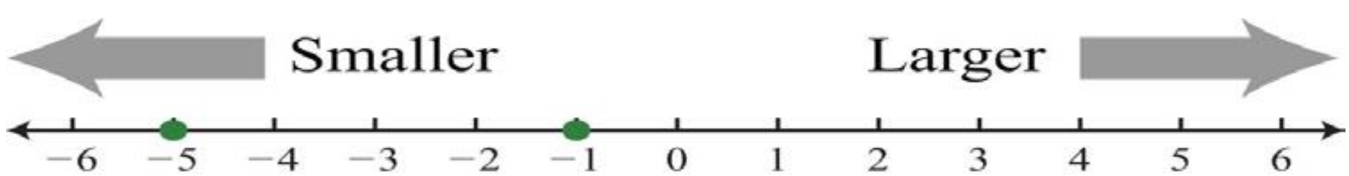
Example 1: Graph the following set of real numbers: {−1, -1/3 , 0, 5/3}.

Solution: Graph the numbers on a number line with a scale where each tick mark represents 1/3 unit.



Ordering Real Numbers

When comparing real numbers on a number line, the larger number will always lie to the right of the smaller one. It is clear that 15 is greater than 5, but it may not be so clear to see that −1 is greater than −5 until we graph each number on a number line.



We use symbols to help us efficiently communicate relationships between numbers on the number line. The symbols used to describe an equality relationship between numbers follow:

$=$ *is equal to*

$\neq$ *is not equal to*

$\approx$ *is approximately equal to*

These symbols are used and interpreted in the following manner:

$$5 = 5 \qquad \textit{5 is equal to 5}$$
$$0 \neq 5 \qquad \textit{0 is not equal to 5}$$
$$\pi \approx 3.14 \qquad \textit{pi is approximately equal to 3.14}$$

We next define symbols that denote an **order relationship** between real numbers.

$<$	*Less than*
$>$	*Greater than*
$\leq$	*Less than or equal to*
$\geq$	*Greater than or equal to*

These symbols allow us to compare two numbers. For example,

$$-120 < -10 \qquad \textit{Negative 120 is less than negative 10.}$$

Since the graph of -120 is to the left of the graph of -10 on the number line, that number is less than -10. We could write an equivalent statement as follows:

$$-10 > -120 \qquad \textit{Negative 10 is greater than negative 120.}$$

Similarly, since the graph of zero is to the right of the graph of any negative number on the number line, zero is greater than any negative number.

$$0 > -50 \qquad \textit{Zero is greater than negative fifty.}$$

The symbols $<$ and $>$ are used to denote strict inequalities, and the symbols $\leq$and $\geq$ are used to denote inclusive inequalities. In some situations, more than one symbol can be correctly applied. For example, the following two statements are both true:

$$-10 < 0 \quad \text{and} \quad -10 \leq 0$$

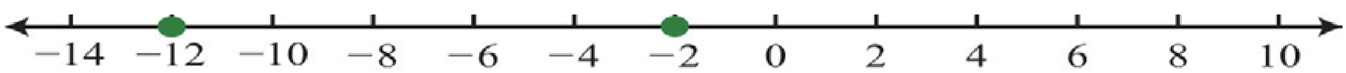
In addition, the "or equal to" component of an inclusive inequality allows us to correctly write the following:

$$-10 \leq -10$$

The logical use of the word “or” requires that only one of the conditions need be true: the “less than” or the “equal to.”

Example 2: Fill in the blank with <, =, or >: −2 ____ −12.

Solution: Use > because the graph of −2 is to the right of the graph of −12 on a number line. Therefore, −2 > −12, which reads “negative two is greater than negative twelve.”



Answer: −2 > −12

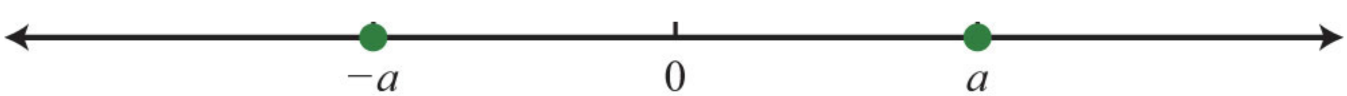
We will often point out the equivalent notation used to express mathematical quantities electronically using the standard symbols available on a keyboard. We begin with the equivalent textual notation for inequalities:

≥	“ >= ”
≤	“ <= ”
≠	“ != “

Many calculators, computer algebra systems, and programming languages use this notation.

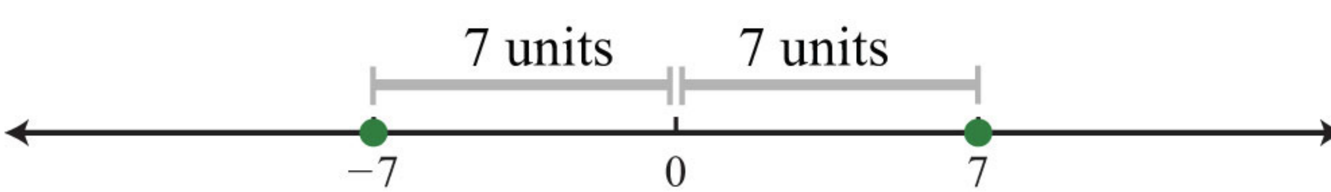
Opposites

The opposite of any real number a is $-a$. Opposite real numbers are the same distance from the origin on a number line, but their graphs lie on opposite sides of the origin and the numbers have opposite signs.



For example, we say that the opposite of 10 is −10.

Next, consider the opposite of a negative number. Given the integer −7, the integer the same distance from the origin and with the opposite sign is +7, or just 7.



Therefore, we say that the opposite of -7 is $-(-7) = 7$. This idea leads to what is often referred to as the double-negative property. For any real number a,

$$-(-a) = a$$

Example 3: What is the opposite of – 3/4?

Solution: Here we apply the double-negative property.

$$-\left(-\frac{3}{4}\right) = \frac{3}{4}$$

Answer: 3/4

Example 4: Simplify: $-(-(4))$.

Solution: Start with the innermost parentheses by finding the opposite of +4.

$$\begin{aligned} -(-(4)) &= -(-(4)) \\ &= -(-4) \\ &= 4 \end{aligned}$$

Answer: 4

Example 5: Simplify: $-(-(-2))$.

Solution: Apply the double-negative property starting with the innermost parentheses.

$$\begin{aligned} -(-(-2)) &= -(-(-2)) \\ &= -(2) \\ &= -2 \end{aligned}$$

Answer: -2

Tip
If there is an even number of consecutive negative signs, then the result is positive. If there is an odd number of consecutive negative signs, then the result is negative.

Try this! Simplify: $-(-(-(5)))$.

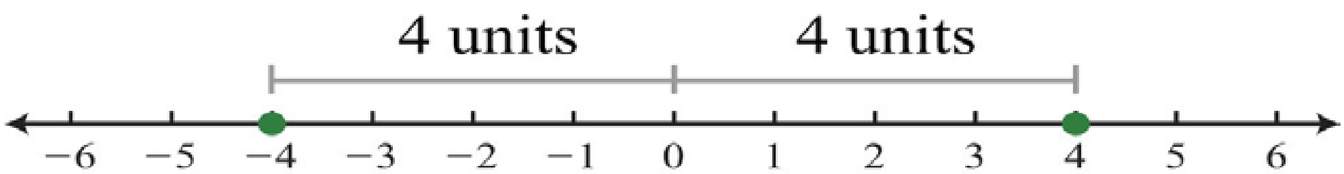
Answer: -5

Absolute Value

The absolute value of a real number a, denoted $|a|$, is defined as the distance between zero (the origin) and the graph of that real number on the number line. Since it is a distance, it is always positive. For example,

$$|-4| = 4 \quad \text{and} \quad |4| = 4$$

Both 4 and −4 are four units from the origin, as illustrated below:



Example 6: Simplify:

a. $|-12|$

b. $|12|$

Solution: Both −12 and 12 are twelve units from the origin on a number line. Therefore,

$$|-12| = 12 \quad \text{and} \quad |12| = 12$$

Answers: a. 12; b. 12

Also, it is worth noting that

$$|0| = 0$$

The absolute value can be expressed textually using the notation abs(a). We often encounter negative absolute values, such as $-|3|$ or −abs(3). Notice that the negative sign is in front of the absolute value symbol. In this case, work the absolute value first and then find the opposite of the result.

$$\begin{array}{c} -|3| \\ \downarrow \\ = -3 \end{array} \quad \text{and} \quad \begin{array}{c} -|-3| \\ \downarrow \\ = -3 \end{array}$$

Try not to confuse this with the double-negative property, which states that $-(-7)=+7$.

Example 7: Simplify: $-|-(-7)|$.

Solution: First, find the opposite of −7 inside the absolute value. Then find the opposite of the result.

$$\begin{aligned} -|-(-7)| &= -|7| \\ &= -7 \end{aligned}$$

Answer: −7

At this point, we can determine what real numbers have a particular absolute value. For example,

$$|\ ?\ | = 5$$

Think of a real number whose distance to the origin is 5 units. There are two solutions: the distance to the right of the origin and the distance to the left of the origin, namely, $\{\pm 5\}$. The symbol (±) is read "plus or minus" and indicates that there are two answers, one positive and one negative.

$$|-5| = 5 \quad \text{and} \quad |5| = 5$$

Now consider the following:

$$|\ ?\ | = -5$$

Here we wish to find a value for which the distance to the origin is negative. Since negative distance is not defined, this equation has no solution. If an equation has no solution, we say the solution is the empty set: Ø.

0.0 Practice – Sets of Numbers

Use set notation to list the described elements.

1. The hours on a clock.

2. The days of the week.

3. The first ten whole numbers.

4. The first ten natural numbers.

5. The first five positive even integers.

6. The first five positive odd integers.

Determine whether the following real numbers are integers, rational, or irrational.

7. ½

8. −3

9. 4.5

10. −5

11. $0.\overline{36}$

12. $0.\overline{3}$

13. 1.001000100001…

14. $1.\overline{001}$

15. $e = 2.71828\ldots$

16. $\sqrt{7} = 2.645751\ldots$

17. −7

18. 3.14

19. $\frac{22}{7}$

20. 1.33

21. 0

22. 8,675,309

True or false.

23. All integers are rational numbers.

24. All integers are whole numbers.

25. All rational numbers are whole numbers.

26. Some irrational numbers are rational.

27. All terminating decimal numbers are rational.

28. All irrational numbers are real.

Choose an appropriate scale and graph the following sets of real numbers on a number line.

29. $\{-3, 0\ 3\}$

30. $\{-2, 2, 4, 6, 8, 10\}$

31. $\{-2, -1/3, 2/3, 5/3\}$

32. $\{-5/2, -1/2, 0, 1/2, 2\}$

33. $\{-5/7, 0, 2/7, 1\}$

34. $\{-5, -2, -1, 0\}$

35. $\{-3, -2, 0, 2, 5\}$

36. $\{-2.5, -1.5, 0, 1, 2.5\}$

37. $\{0, 0.3, 0.6, 0.9, 1.2\}$

38. $\{-10, 30, 50\}$

39. $\{-6, 0, 3, 9, 12\}$

40. $\{-15, -9, 0, 9, 15\}$

Fill in the blank with <, =, or >.

41. -7 ___ 0

42. 30 ___ 2

43. 10 ___ -10

44. -150 ___ -75

45. -0.5 ___ -1.5

46. 0 ___ 0

47. -500 ___ 200

48. -1 ___ -200

49. -10 ___ -10

50. -40 ___ -41

True or false.

51. $5 \neq 7$

52. $4 = 5$

53. $1 \neq 1$

54. $-5 > -10$

55. $4 \leq 4$

56. $-12 \geq 0$

57. $-10 = -10$

58. $3 > 3$

59. $-1000 < -20$

60. $0 = 0$

61. List three integers less than -5.

62. List three integers greater than -10.

63. List three rational numbers less than zero.

64. List three rational numbers greater than zero.

65. List three integers between -20 and -5.

66. List three rational numbers between 0 and 1.

Translate each statement into an English sentence.

67. $10<20$

68. $-50\leq-10$

69. $-4\neq0$

70. $30\geq-1$

71. $0=0$

72. $e \approx 2.718$

Translate the following into a mathematical statement.

73. Negative seven is less than zero.

74. Twenty-four is not equal to ten.

75. Zero is greater than or equal to negative one.

76. Four is greater than or equal to negative twenty-one.

77. Negative two is equal to negative two.

78. Negative two thousand is less than negative one thousand.

Simplify.

79. $-(-9)$

80. $-(-3/5)$

81. $-(10)$

82. $-(3)$

83. $-(5)$

84. $-(3/4)$

85. $-(-1)$

86. $-(-(-1))$

87. $-(-(1))$

88. $-(-(-3))$

89. $-(-(-(-11)))$

90. What is the opposite of $-1/2$

91. What is the opposite of π?

92. What is the opposite -0.01?

93. Is the opposite of -12 smaller or larger than -11?

94. Is the opposite of 7 smaller or larger than -6?

Fill in the blank with <, =, or >.

95. -7 ___ $-(-8)$

96. 6 ___ $-(6)$

97. 13 ___ $-(-12)$

98. $-(-5)$ ___ $-(-2)$

99. -100 ___ $-(-(-50))$

100. 44 ___ $-(-44)$

Simplify.

101. $|20|$

102. $|-20|$

103. $|-33|$

104. $|-0.75|$

105. $|-2/5|$

106. $|3/8|$

107. $|0|$

108. $|1|$

109. $-|12|$

110. $-|-20|$

111. $-|20|$

112. $-|-8|$

113. $-|7|$

114. $-|-3/16|$

115. $-(-|8/9|)$

116. $|-(-2)|$

117. $-|-(-3)|$

118. $-(-|5|)$

119. $-(-|-45|)$

120. $-|-(-21)|$

121. abs(6)

122. abs(−7)

123. −abs(5)

124. $-\text{abs}(-19)$

125. $-(-\text{abs}(9))$

126. $-\text{abs}(-(-12))$

Determine the unknown.

127. $|?|=9$

128. $|?|=15$

129. $|?|=0$

130. $|?|=1$

131. $|?|=-8$

132. $|?|=-20$

133. $|?|-10=-2$

134. $|?|+5=14$

Fill in the blank with <, =, or >.

135. $|-2|$ ____ 0

136. $|-7|$ ____ $|-10|$

137. -10 ____ $-|-2|$

138. $|-6|$ ____ $|-(-6)|$

139. $-|3|$ ____ $|-(-5)|$

140. 0 ____ $-|-(-4)|$

Discussion Topics

141. Research and discuss the history of the number zero.

142. Research and discuss the various numbering systems throughout history.

143. Research and discuss the definition and history of π.

144. Research the history of irrational numbers. Who is credited with proving that the square root of 2 is irrational and what happened to him?

145. Research and discuss the history of absolute value.

146. Discuss the “just make it positive” definition of absolute value.

0.1

Pre-Algebra - Integers

Objective: Add, Subtract, Multiply and Divide Positive and Negative Numbers.

The ability to work comfortably with negative numbers is essential to success in algebra. For this reason we will do a quick review of adding, subtracting, multiplying and dividing of integers. **Integers** are all the positive whole numbers, zero, and their opposites (negatives). As this is intended to be a review of integers, the descriptions and examples will not be as detailed as a normal lesson.

World View Note: The first set of rules for working with negative numbers was written out by the Indian mathematician Brahmagupta.

When adding integers we have two cases to consider. The first is if the signs match, both positive or both negative. If the signs match we will add the numbers together and keep the sign. This is illustrated in the following examples

Example 1.

$$\begin{array}{rl} -5+(-3) & \text{Same sign, add } 5+3\text{, keep the negative} \\ -8 & \text{Our Solution} \end{array}$$

Example 2.

$$\begin{array}{rl} -7+(-5) & \text{Same sign, add } 7+5\text{, keep the negative} \\ -12 & \text{Our Solution} \end{array}$$

If the signs don't match, one positive and one negative number, we will subtract the numbers (as if they were all positive) and then use the sign from the larger number. This means if the larger number is positive, the answer is positive. If the larger number is negative, the answer is negative. This is shown in the following examples.

Example 3.

$$\begin{array}{rl} -7+2 & \text{Different signs, subtract } 7-2\text{, use sign from bigger number, negative} \\ -5 & \text{Our Solution} \end{array}$$

Example 4.

$$\begin{array}{rl} -4+6 & \text{Different signs, subtract } 6-4\text{, use sign from bigger number, positive} \\ 2 & \text{Our Solution} \end{array}$$

Example 5.

$4+(-3)$ Different signs, subtract $4-3$, use sign from bigger number, positive
1 Our Solution

Example 6.

$7+(-10)$ Different signs, subtract $10-7$, use sign from bigger number, negative
-3 Our Solution

For subtraction of negatives we will change the problem to an addition problem which we can then solve using the above methods. The way we change a subtraction to an addition is to add the opposite of the number after the subtraction sign. Often this method is refered to as "add the opposite." This is illustrated in the following examples.

Example 7.

$8-3$ Add the opposite of 3
$8+(-3)$ Different signs, subtract $8-3$, use sign from bigger number, positive
5 Our Solution

Example 8.

$-4-6$ Add the opposite of 6
$-4+(-6)$ Same sign, add $4+6$, keep the negative
-10 Our Solution

Example 9.

$9-(-4)$ Add the opposite of -4
$9+4$ Same sign, add $9+4$, keep the positive
13 Our Solution

Example 10.

$-6-(-2)$ Add the opposite of -2
$-6+2$ Different sign, subtract $6-2$, use sign from bigger number, negative
-4 Our Solution

Multiplication and division of integers both work in a very similar pattern. The short description of the process is we multiply and divide like normal, if the signs match (both positive or both negative) the answer is positive. If the signs don't match (one positive and one negative) then the answer is negative. This is shown in the following examples

Example 11.

$$(4)(-6) \quad \text{Signs do not match, answer is negative}$$
$$-24 \quad \text{Our Solution}$$

Example 12.

$$\frac{-36}{-9} \quad \text{Signs match, answer is positive}$$
$$4 \quad \text{Our Solution}$$

Example 13.

$$-2(-6) \quad \text{Signs match, answer is positive}$$
$$12 \quad \text{Our Solution}$$

Example 14.

$$\frac{15}{-3} \quad \text{Signs do not match, answer is negative}$$
$$-5 \quad \text{Our Solution}$$

A few things to be careful of when working with integers. First be sure not to confuse a problem like $-3-8$ with $-3(-8)$. The second problem is a multiplication problem because there is nothing between the 3 and the parenthesis. If there is no operation written in between the parts, then we assume that means we are multiplying. The $-3-8$ problem, is subtraction because the subtraction separates the 3 from what comes after it. Another item to watch out for is to be careful not to mix up the pattern for adding and subtracting integers with the pattern for multiplying and dividing integers. They can look very similar, for example if the signs match on addition, the we keep the negative, $-3+(-7)=-10$, but if the signs match on multiplication, the answer is positive, $(-3)(-7)=21$.

0.1 Practice - Integers

Evaluate each expression.

1) $1-3$

2) $4-(-1)$

3) $(-6)-(-8)$

4) $(-6)+8$

5) $(-3)-3$

6) $(-8)-(-3)$

7) $3-(-5)$

8) $7-7$

9) $(-7)-(-5)$

10) $(-4)+(-1)$

11) $3-(-1)$

12) $(-1)+(-6)$

13) $6-3$

14) $(-8)+(-1)$

15) $(-5)+3$

16) $(-1)-8$

17) $2-3$

18) $5-7$

19) $(-8)-(-5)$

20) $(-5)+7$

21) $(-2)+(-5)$

22) $1+(-1)$

23) $5-(-6)$

24) $8-(-1)$

25) $(-6)+3$

26) $(-3)+(-1)$

27) $4-7$

28) $7-3$

29) $(-7)+7$

30) $(-3)+(-5)$

Find each product.

31) $(4)(-1)$

32) $(7)(-5)$

33) $(10)(-8)$

34) $(-7)(-2)$

35) $(-4)(-2)$

36) $(-6)(-1)$

37) $(-7)(8)$

38) $(6)(-1)$

39) $(9)(-4)$

40) $(-9)(-7)$

41) $(-5)(2)$

42) $(-2)(-2)$

43) $(-5)(4)$

44) $(-3)(-9)$

45) $(4)(-6)$

Find each quotient.

46) $\frac{30}{-10}$

47) $\frac{-49}{-7}$

48) $\frac{-12}{-4}$

49) $\frac{-2}{-1}$

50) $\frac{30}{6}$

51) $\frac{20}{10}$

52) $\frac{27}{3}$

53) $\frac{-35}{-5}$

54) $\frac{80}{-8}$

55) $\frac{-8}{-2}$

56) $\frac{50}{5}$

57) $\frac{-16}{2}$

58) $\frac{48}{8}$

59) $\frac{60}{-10}$

60) $\frac{54}{-6}$

0.2

Pre-Algebra - Fractions

Objective: Reduce, add, subtract, multiply, and divide with fractions.

Working with fractions is a very important foundation to algebra. Here we will briefly review reducing, multiplying, dividing, adding, and subtracting fractions. As this is a review, concepts will not be explained in detail as other lessons are.

World View Note: The earliest known use of fraction comes from the Middle Kingdom of Egypt around 2000 BC!

We always like our final answers when working with fractions to be reduced. Reducing fractions is simply done by dividing both the numerator and denominator by the same number. This is shown in the following example

Example 15.

$$\frac{36}{84} \quad \text{Both numerator and denominator are divisible by 4}$$

$$\frac{36 \div 4}{84 \div 4} = \frac{9}{21} \quad \text{Both numerator and denominator are still divisible by 3}$$

$$\frac{9\div 3}{21\div 3}=\frac{3}{7}\quad \text{Our Soultion}$$

The previous example could have been done in one step by dividing both numerator and denominator by 12. We also could have divided by 2 twice and then divided by 3 once (in any order). It is not important which method we use as long as we continue reducing our fraction until it cannot be reduced any further.

The easiest operation with fractions is multiplication. We can multiply fractions by multiplying straight across, multiplying numerators together and denominators together.

Example 16.

$$\frac{6}{7}\cdot\frac{3}{5}\quad \text{Multiply numerators across and denominators across}$$

$$\frac{18}{35}\quad \text{Our Solution}$$

When multiplying we can reduce our fractions before we multiply. We can either reduce vertically with a single fraction, or diagonally with several fractions, as long as we use one number from the numerator and one number from the denominator.

Example 17.

$$\frac{25}{24}\cdot\frac{32}{55}\quad \text{Reduce 25 and 55 by dividing by 5. Reduce 32 and 24 by dividing by 8}$$

$$\frac{5}{3}\cdot\frac{4}{11}\quad \text{Multiply numerators across and denominators across}$$

$$\frac{20}{33}\quad \text{Our Solution}$$

Dividing fractions is very similar to multiplying with one extra step. Dividing fractions requires us to first take the reciprocal of the second fraction and multiply. Once we do this, the multiplication problem solves just as the previous problem.

Example 18.

$$\frac{21}{16}\div\frac{28}{6}\quad \text{Multiply by the reciprocal}$$

$$\frac{21}{16}\cdot\frac{6}{28}\quad \text{Reduce 21 and 28 by dividing by 7. Reduce 6 and 16 by dividing by 2}$$

$$\frac{3}{8}\cdot\frac{3}{4}\quad \text{Multiply numerators across and denominators across}$$

$$\frac{9}{32}\quad \text{Our Soultion}$$

To add and subtract fractions we will first have to find the least common denominator (LCD). There are several ways to find an LCD. One way is to find the smallest multiple of the largest denominator that you can also divide the small denomiator by.

Example 19.

$$\begin{aligned} \text{Find the LCD of 8 and 12} &\quad \text{Test multiples of 12} \\ 12?\,\frac{12}{8} &\quad \text{Can}'t\text{ divide 12 by 8} \\ 24?\,\frac{24}{8}=3 &\quad \text{Yes! We can divide 24 by 8!} \\ 24 &\quad \text{Our Soultion} \end{aligned}$$

Adding and subtracting fractions is identical in process. If both fractions already have a common denominator we just add or subtract the numerators and keep the denominator.

Example 20.

$$\frac{7}{8}+\frac{3}{8}\quad \text{Same denominator, add numerators } 7+3$$

$$\frac{10}{8}\quad \text{Reduce answer, dividing by 2}$$

$$\frac{5}{4}\quad \text{Our Solution}$$

While $\frac{5}{4}$ can be written as the mixed number $1\frac{1}{4}$, in algebra we will almost never use mixed numbers. For this reason we will always use the improper fraction, not the mixed number.

Example 21.

$$\frac{13}{6} - \frac{9}{6} \quad \text{Same denominator, subtract numerators } 13 - 9$$

$$\frac{4}{6} \quad \text{Reduce answer, dividing by } 2$$

$$\frac{2}{3} \quad \text{Our Solution}$$

If the denominators do not match we will first have to identify the LCD and build up each fraction by multiplying the numerators and denominators by the same number so the denominator is built up to the LCD.

Example 22.

$$\frac{5}{6} + \frac{4}{9} \quad \text{LCD is 18.}$$

$$\frac{\mathbf{3} \cdot 5}{\mathbf{3} \cdot 6} + \frac{4 \cdot \mathbf{2}}{9 \cdot \mathbf{2}} \quad \text{Multiply first fraction by 3 and the second by 2}$$

$$\frac{15}{18} + \frac{8}{18} \quad \text{Same denominator, add numerators, } 15 + 8$$

$$\frac{23}{18} \quad \text{Our Solution}$$

Example 23.

$$\frac{2}{3} - \frac{1}{6} \quad \text{LCD is 6}$$

$$\frac{2 \cdot 2}{2 \cdot 3} - \frac{1}{6} \quad \text{Multiply first fraction by 2, the second already has } a \text{ denominator of 6}$$

$$\frac{4}{6} - \frac{1}{6} \quad \text{Same denominator, subtract numerators, } 4 - 1$$

$$\frac{3}{6} \quad \text{Reduce answer, dividing by 3}$$

$$\frac{1}{2} \quad \text{Our Solution}$$

0.2 Practice - Fractions

Simplify each. Leave your answer as an improper fraction.

1) $\frac{42}{12}$

2) $\frac{25}{20}$

3) $\frac{35}{25}$

4) $\frac{24}{9}$

5) $\frac{54}{36}$

6) $\frac{30}{24}$

7) $\frac{45}{36}$

8) $\frac{36}{27}$

9) $\frac{27}{18}$

10) $\frac{48}{18}$

11) $\frac{40}{16}$

12) $\frac{48}{42}$

13) $\frac{63}{18}$

14) $\frac{16}{12}$

15) $\frac{80}{60}$

16) $\frac{72}{48}$

17) $\frac{72}{60}$

18) $\frac{126}{108}$

19) $\frac{36}{24}$

20) $\frac{160}{140}$

Find each product.

21) $(9)(\frac{8}{9})$

22) $(-2)(-\frac{5}{6})$

23) $(2)(-\frac{2}{9})$

24) $(-2)(\frac{1}{3})$

25) $(-2)(\frac{13}{8})$

26) $(\frac{3}{2})(\frac{1}{2})$

27) $(-\frac{6}{5})(-\frac{11}{8})$

28) $(-\frac{3}{7})(-\frac{11}{8})$

29) $(8)(\frac{1}{2})$

30) $(-2)(-\frac{9}{7})$

31) $(\frac{2}{3})(\frac{3}{4})$

32) $(-\frac{17}{9})(-\frac{3}{5})$

33 $(2)(\frac{3}{2})$

34) $(\frac{17}{9})(-\frac{3}{5})$

35) $(\frac{1}{2})(-\frac{7}{5})$

36) $(\frac{1}{2})(\frac{5}{7})$

Find each quotient.

37) $-2 \div \frac{7}{4}$

38) $\frac{-12}{7} \div \frac{-9}{5}$

39) $\frac{-1}{9} \div \frac{-1}{2}$

40) $-2 \div \frac{-3}{2}$

41) $\frac{-3}{2} \div \frac{13}{7}$

42) $\frac{5}{3} \div \frac{7}{5}$

43) $-1 \div \frac{2}{3}$

44) $\frac{10}{9} \div -6$

45) $\frac{8}{9} \div \frac{1}{5}$

46) $\frac{1}{6} \div \frac{-5}{3}$

47) $\frac{-9}{7} \div \frac{1}{5}$

48) $\frac{-13}{8} \div \frac{-15}{8}$

49) $\frac{-2}{9} \div \frac{-3}{2}$

50) $\frac{-4}{5} \div \frac{-13}{8}$

51) $\frac{1}{10} \div \frac{3}{2}$

52) $\frac{5}{3} \div \frac{5}{3}$

Evaluate each expression.

53) $\frac{1}{3} + (-\frac{4}{3})$

54) $\frac{1}{7} + (-\frac{11}{7})$

55) $\frac{3}{7} - \frac{1}{7}$

56) $\frac{1}{3} + \frac{5}{3}$

57) $\frac{11}{6} + \frac{7}{6}$

58) $(-2) + (-\frac{15}{8})$

59) $\frac{3}{5} + \frac{5}{4}$

60) $(-1) - \frac{2}{3}$

61) $\frac{2}{5} + \frac{5}{4}$

62) $\frac{12}{7} - \frac{9}{7}$

63) $\frac{9}{8} + (-\frac{2}{7})$

64) $(-2) + \frac{5}{6}$

65) $1 + (-\frac{1}{3})$

66) $\frac{1}{2} - \frac{11}{6}$

67) $(-\frac{1}{2}) + \frac{3}{2}$

68) $\frac{11}{8} - \frac{1}{2}$

69) $\frac{1}{5} + \frac{3}{4}$

70) $\frac{6}{5} - \frac{8}{5}$

71) $(-\frac{5}{7}) - \frac{15}{8}$

72) $(-\frac{1}{3}) + (-\frac{8}{5})$

73) $6 - \frac{8}{7}$

74) $(-6) + (-\frac{5}{3})$

75) $\frac{3}{2} - \frac{15}{8}$

76) $(-1) - (-\frac{1}{3})$

77) $(-\frac{15}{8}) + \frac{5}{3}$

78) $\frac{3}{2} + \frac{9}{7}$

79) $(-1) - (-\frac{1}{6})$

80) $(-\frac{1}{2}) - (-\frac{3}{5})$

81) $\frac{5}{3} - (-\frac{1}{3})$

82) $\frac{9}{7} - (-\frac{5}{3})$

0.3

Pre-Algebra - Order of Operations

Objective: Evaluate expressions using the order of operations, including the use of absolute value.

When simplifying expressions it is important that we simplify them in the correct order. Consider the following problem done two different ways:

Example 24.

$$\begin{array}{cc} \underbrace{2+5}\cdot 3 & \text{Add First} \\ \underbrace{7\cdot 3} & \text{Multiply} \\ 21 & \text{Solution} \end{array} \qquad \begin{array}{cc} 2+\underbrace{5\cdot 3} & \text{Multiply} \\ \underbrace{2+15} & \text{Add} \\ 17 & \text{Solution} \end{array}$$

The previous example illustrates that if the same problem is done two different ways we will arrive at two different solutions. However, only one method can be correct. It turns out the second method, 17, is the correct method. The order of operations ends with the most basic of operations, addition (or subtraction). Before addition is completed we must do repeated addition or multiplication (or division). Before multiplication is completed we must do repeated multiplication or exponents. When we want to do something out of order and make it come first we will put it in parenthesis (or grouping symbols). This list then is our order of operations we will use to simplify expressions.

Order of Operations:

Parenthesis (Grouping)

Exponents

Multiply and Divide (Left to Right)

Add and Subtract (Left to Right)

Multiply and Divide are on the same level because they are the same operation (division is just multiplying by the reciprocal). This means they must be done left to right, so some problems we will divide first, others we will multiply first. The same is true for adding and subtracting (subtracting is just adding the opposite).

Often students use the word PEMDAS to remember the order of operations, as the first letter of each operation creates the word PEMDAS. However, it is the author's suggestion to think about PEMDAS as a vertical word written as:

P
E
MD
AS

so we don't forget that multiplication and division are done left to right (same with addition and subtraction). Another way students remember the order of operations is to think of a phrase such as "Please Excuse My Dear Aunt Sally" where each word starts with the same letters as the order of operations start with.

World View Note: The first use of grouping symbols are found in 1646 in the Dutch mathematician, Franciscus van Schooten's text, Vieta. He used a bar over the expression that is to be evaluated first. So problems like $2(3+5)$ were written as $2\cdot\overline{3+5}$.

Example 25.

$$\begin{array}{rl}
2+3(\underbrace{9-4})^2 & \text{Parenthesis first}\\
2+3\underbrace{(5)^2} & \text{Exponents}\\
2+\underbrace{3(25)} & \text{Multiply}\\
\underbrace{2+75} & \text{Add}\\
77 & \text{Our Solution}
\end{array}$$

It is very important to remember to multiply and divide from from left to right!

Example 26.

$$\begin{array}{rl}
\underbrace{30\div 3}\cdot 2 & \text{Divide first (left to right!)}\\
\underbrace{10\cdot 2} & \text{Multiply}\\
20 & \text{Our Solution}
\end{array}$$

In the previous example, if we had multiplied first, five would have been the answer which is incorrect.

If there are several parenthesis in a problem we will start with the inner most parenthesis and work our way out. Inside each parenthesis we simplify using the order of operations as well. To make it easier to know which parenthesis goes with which parenthesis, different types of parenthesis will be used such as { } and [ ] and ( ), these parenthesis all mean the same thing, they are parenthesis and must be evaluated first.

Example 27.

$$\begin{array}{rl}
2\{8^2-7[32-4(\underbrace{3^2}+1)](-1)\} & \text{Inner most parenthesis, exponents first}\\
2\{8^2-7[32-4(\underbrace{9+1})](-1)\} & \text{Add inside those parenthesis}\\
2\{8^2-7[32-\underbrace{4(10)}](-1)\} & \text{Multiply inside inner most parenthesis}\\
2\{8^2-7[\underbrace{32-40}](-1)\} & \text{Subtract inside those parenthesis}\\
2\{\underbrace{8^2}-7[-8](-1)\} & \text{Exponents next}\\
2\{64\underbrace{-7[-8]}(-1)\} & \text{Multiply left to right, sign with the number}\\
2\{64+\underbrace{56(-1)}\} & \text{Finish multiplying}\\
2\{\underbrace{64-56}\} & \text{Subtract inside parenthesis}\\
\underbrace{2\{8\}} & \text{Multiply}\\
16 & \text{Our Solution}
\end{array}$$

As the above example illustrates, it can take several steps to complete a problem. The key to successfully solve order of operations problems is to take the time to show your work and do one step at a time. This will reduce the chance of making a mistake along the way.

There are several types of grouping symbols that can be used besides parenthesis. One type is a fraction bar. If we have a fraction, the entire numerator and the entire denominator must be evaluated before we reduce the fraction. In these cases we can simplify in both the numerator and denominator at the same time.

Example 28.

$$\frac{2^4 - (-8)\cdot 3}{\underbrace{15 \div 5} - 1} \quad \text{Exponent in the numerator, divide in denominator}$$

$$\frac{16 - \overbrace{(-8)\cdot 3}}{\underbrace{3 - 1}} \quad \text{Multiply in the numerator, subtract in denominator}$$

$$\frac{\overbrace{16 - (-24)}}{2} \quad \text{Add the opposite to simplify numerator, denominator is done.}$$

$$\frac{40}{2} \quad \text{Reduce, divide}$$

$$20 \quad \text{Our Solution}$$

Another type of grouping symbol that also has an operation with it, absolute value. When we have absolute value we will evaluate everything inside the absolute value, just as if it were a normal parenthesis. Then once the inside is completed we will take the absolute value, or distance from zero, to make the number positive.

Example 29.

$$\begin{array}{rl}
1 + 3|-4^2 - (-8)| + 2|3 + \underbrace{(-5)^2}| & \text{Evaluate absolute values first, exponents} \\
1 + 3|\underbrace{-16 - (-8)}| + 2|\underbrace{3 + 25}| & \text{Add inside absolute values} \\
1 + 3|\underbrace{-8}| + 2|\underbrace{28}| & \text{Evaluate absolute values} \\
1 + \underbrace{3(8)} + 2(28) & \text{Multiply left to right} \\
1 + 24 + \underbrace{2(28)} & \text{Finish multiplying} \\
\underbrace{1 + 24} + 56 & \text{Add left to right} \\
\underbrace{25 + 56} & \text{Add} \\
81 & \text{Our Solution}
\end{array}$$

The above example also illustrates an important point about exponents. Exponents only are considered to be on the number they are attached to. This means when we see -4^2, only the 4 is squared, giving us $-(4^2)$ or -16. But when the negative is in parentheses, such as $(-5)^2$ the negative is part of the number and is also squared giving us a positive solution, 25.

0.3 Practice - Order of Operation

Solve.

1) $-6\cdot 4(-1)$

2) $(-6\div 6)^3$

3) $3+(8)\div |4|$

4) $5(-5+6)\cdot 6^2$

5) $8\div 4\cdot 2$

6) $7-5+6$

7) $[-9-(2-5)]\div(-6)$

8) $(-2\cdot 2^3\cdot 2)\div(-4)$

9) $-6+(-3-3)^2\div\left|3\right|$

10) $(-7-5)\div[-2-2-(-6)]$

11) $4-2\left|3^2-16\right|$

12) $\frac{-10-6}{(-2)^2}-5$

13) $[-1-(-5)]|3+2|$

14) $-3-\{3-[-3(2+4)-(-2)]\}$

15) $\frac{2+4|7+2^2|}{4\cdot 2+5\cdot 3}$

16) $-4-[2+4(-6)-4-\left|2^2-5\cdot 2\right|]$

17) $[6\cdot 2+2-(-6)](-5+\left|\frac{-18}{6}\right|)$

18) $2\cdot(-3)+3-6[-2-(-1-3)]$

19) $\frac{-13-2}{2-(-1)^3+(-6)-[-1-(-3)]}$

20) $\frac{-5^2+(-5)^2}{|4^2-2^5|-2\cdot 3}$

21) $6\cdot\frac{-8-4+(-4)-[-4-(-3)]}{(4^2+3^2)\div 5}$

22) $\frac{-9\cdot 2-(3-6)}{1-(-2+1)-(-3)}$

23) $\frac{2^3+4}{-18-6+(-4)-[-5(-1)(-5)]}$

24) $\frac{13+(-3)^2+4(-3)+1-[-10-(-6)]}{\{[4+5]\div[4^2-3^2(4-3)-8]\}+12}$

25) $\frac{5+3^2-24\div 6\cdot 2}{[5+3(2^2-5)]+|2^2-5|^2}$

0.4

Pre-Algebra - Properties of Algebra

Objective: Simplify algebraic expressions by substituting given values, distributing, and combining like terms

In algebra we will often need to simplify an expression to make it easier to use. There are three basic forms of simplifying which we will review here.

World View Note: The term "Algebra" comes from the Arabic word al-jabr which means "reunion". It was first used in Iraq in 830 AD by Mohammad ibn-Musa al-Khwarizmi.

The first form of simplifying expressions is used when we know what number each variable in the expression represents. If we know what they represent we can replace each variable with the equivalent number and simplify what remains using order of operations.

Example 30.

$$\begin{array}{rl} p(q+6) \text{ when } p=3 \text{ and } q=5 & \text{Replace } p \text{ with 3 and } q \text{ with 5} \\ (3)\mathbf{((5)+6)} & \text{Evaluate parenthesis} \\ \mathbf{(3)(11)} & \text{Multiply} \\ 33 & \text{Our Solution} \end{array}$$

Whenever a variable is replaced with something, we will put the new number inside a set of parenthesis. Notice the 3 and 5 in the previous example are in parenthesis. This is to preserve operations that are sometimes lost in a simple replacement. Sometimes the parenthesis won't make a difference, but it is a good habbit to always use them to prevent problems later.

Example 31.

$$\begin{array}{rl} x+zx(3-z)\left(\frac{x}{3}\right) \text{ when } x=-6 \text{ and } z=-2 & \text{Replace all } x's \text{ with 6 and } z's \text{ with 2} \\ (-6)+(-2)(-6)\mathbf{(3-(-2))}\left(\frac{\mathbf{(-6)}}{\mathbf{3}}\right) & \text{Evaluate parenthesis} \\ -6+\mathbf{(-2)(-6)}(5)(-2) & \text{Multiply left to right} \\ -6+\mathbf{12(5)}(-2) & \text{Multiply left to right} \\ -6+\mathbf{60(-2)} & \text{Multiply} \\ \mathbf{-6-120} & \text{Subtract} \\ -126 & \text{Our Solution} \end{array}$$

It will be more common in our study of algebra that we do not know the value of the variables. In this case, we will have to simplify what we can and leave the variables in our final solution. One way we can simplify expressions is to combine like terms. **Like terms** are terms where the variables match exactly (exponents included). Examples of like terms would be $3xy$ and $-7xy$ or $3a^2b$ and $8a^2b$ or -3 and 5. If we have like terms we are allowed to add (or subtract) the numbers in front of the variables, then keep the variables the same. This is shown in the following examples

Example 32.

$$\begin{array}{rl} 5x-2y-8x+7y & \text{Combine like terms } 5x-8x \text{ and } -2y+7y \\ -3x+5y & \text{Our Solution} \end{array}$$

Example 33.

$$\begin{array}{rl} 8x^2-3x+7-2x^2+4x-3 & \text{Combine like terms } 8x^2-2x^2 \text{ and } -3x+4x \text{ and } 7-3 \\ 6x^2+x+4 & \text{Our Solution} \end{array}$$

As we combine like terms we need to interpret subtraction signs as part of the following term. This means if we see a subtraction sign, we treat the following term like a negative term, the sign always stays with the term.

A final method to simplify is known as distributing. Often as we work with problems there will be a set of parenthesis that make solving a problem difficult, if not impossible. To get rid of these unwanted parenthesis we have the distributive property. Using this property we multiply the number in front of the parenthesis by each term inside of the parenthesis.

$$\textbf{Distributive Property: } \boldsymbol{a(b+c)} = \mathbf{ab}+\mathbf{ac}$$

Several examples of using the distributive property are given below.

Example 34.

$$\begin{array}{rl} 4(2x-7) & \text{Multiply each term by } 4 \\ 8x-28 & \text{Our Solution} \end{array}$$

Example 35.

$$\begin{array}{rl} -7(5x-6) & \text{Multiply each term by } -7 \\ -35+42 & \text{Our Solution} \end{array}$$

In the previous example we again use the fact that the sign goes with the number, this means we treat the -6 as a negative number, this gives $(-7)(-6)=42$, a positive number. The most common error in distributing is a sign error, be very careful with your signs!

It is possible to distribute just a negative through parenthesis. If we have a negative in front of parenthesis we can think of it like a -1 in front and distribute the -1 through. This is shown in the following example.

Example 36.

$$\begin{array}{rl} -(4x-5y+6) & \text{Negative can be thought of as } -1 \\ -1(4x-5y+6) & \text{Multiply each term by } -1 \\ -4x+5y-6 & \text{Our Solution} \end{array}$$

Distributing through parenthesis and combining like terms can be combined into one problem. Order of operations tells us to multiply (distribute) first then add or subtract last (combine like terms). Thus we do each problem in two steps, distribute then combine.

Example 37.

$$\begin{array}{rl} 5+3(2x-4) & \text{Distribute 3, multipling each term} \\ 5+6x-12 & \text{Combine like terms } 5-12 \\ -7+6x & \text{Our Solution} \end{array}$$

Example 38.

$$\begin{array}{rl} 3x-2(4x-5) & \text{Distribute } -2\text{, multilpying each term} \\ 3x-8x+10 & \text{Combine like terms } 3x-8x \\ -5x+10 & \text{Our Solution} \end{array}$$

In the previous example we distributed -2, not just 2. This is because we will always treat subtraction like a negative sign that goes with the number after it. This makes a big difference when we multiply by the -5 inside the parenthesis, we now have a positive answer. Following are more involved examples of distributing and combining like terms.

Example 39.

$$\begin{array}{rl} 2(5x-8)-6(4x+3) & \text{Distribute 2 into first parenthesis and } -6 \text{ into second} \\ 10x-16-24x-18 & \text{Combine like terms } 10x-24x \text{ and } -16-18 \\ -14x-34 & \text{Our Solution} \end{array}$$

Example 40.

$$\begin{array}{rl} 4(3x-8)-(2x-7) & \text{Negative (subtract) in middle can be thought of as } -1 \\ 4(3x-8)-1(2x-7) & \text{Distribute 4 into first parenthesis, } -1 \text{ into second} \\ 12x-32-2x+7 & \text{Combine like terms } 12x-2x \text{ and } -32+7 \\ 10x-25 & \text{Our Solution} \end{array}$$

0.4 Practice - Properties of Algebra

Evaluate each using the values given.

1) $p+1+q-m$; use $m=1, p=3, q=4$

2) y^2+y-z; use $y=5, z=1$

3) $p-\frac{pq}{6}$; use $p=6$ and $q=5$

4) $\frac{6+z-y}{3}$; use $y=1, z=4$

5) $c^2-(a-1)$; use $a=3$ and $c=5$

6) $x+6z-4y$; use $x=6, y=4, z=4$

7) $5j+\frac{kh}{2}$; use $h=5, j=4, k=2$

8) $5(b+a)+1+c$; use $a=2, b=6, c=5$

9) $\frac{4-(p-m)}{2}+q$; use $m=4, p=6, q=6$

10) $z+x-(1^2)^3$; use $x=5, z=4$

11) $m+n+m+\frac{n}{2}$; use $m=1$ and $n=2$

12) $3+z-1+y-1$; use $y=5, z=4$

13) $q-p-(q-1-3)$; use $p=3, q=6$

14) $p+(q-r)(6-p)$; use $p=6, q=5, r=5$

15) $y-[4-y-(z-x)]$; use $x=3, y=1, z=6$

16) $4z-(x+x-(z-z))$; use $x=3, z=2$

17) $k\times 3^2-(j+k)-5$; use $j=4, k=5$

18) $a^3(c^2-c)$; use $a=3, c=2$

19) $zx-(z-\frac{4+x}{6})$; use $x=2, z=6$

20) $5+qp+pq-q$; use $p=6, q=3$

Combine Like Terms

21) $r-9+10$

22) $-4x+2-4$

23) $n+n$

24) $4b+6+1+7b$

25) $8v+7v$

26) $-x+8x$

27) $-7x-2x$

28) $-7a-6+5$

29) $k-2+7$

30) $-8p+5p$

31) $x-10-6x+1$

32) $1-10n-10$

33) $m-2m$

34) $1-r-6$

35) $9n-1+n+4$

36) $-4b+9b$

Distribute

37) $-8(x-4)$

38) $3(8v+9)$

39) $8n(n+9)$

40) $-(-5+9a)$

41) $7k(-k+6)$

42) $10x(1+2x)$

43) $-6(1+6x)$

44) $-2(n+1)$

45) $8m(5-m)$

46) $-2p(9p-1)$

47) $-9x(4-x)$

48) $4(8n-2)$

49) $-9b(b-10)$

50) $-4(1+7r)$

51) $-8n(5+10n)$

52) $2x(8x-10)$

Simplify.

53) $9(b+10)+5b$

54) $4v-7(1-8v)$

55) $-3x(1-4x)-4x^2$

56) $-8x+9(-9x+9)$

57) $-4k^2-8k\,(8k+1)$

58) $-9-10(1+9a)$

59) $1-7(5+7p)$

60) $-10(x-2)-3$

61) $-10-4(n-5)$

62) $-6(5-m)+3m$

63) $4(x+7)+8(x+4)$

64) $-2r(1+4r)+8r(-r+4)$

65) $-8(n+6)-8n(n+8)$

66) $9(6b+5)-4b(b+3)$

67) $7(7+3v)+10(3-10v)$

68) $-7(4x-6)+2(10x-10)$

69) $2n(-10n+5)-7(6-10n)$

70) $-3(4+a)+6a(9a+10)$

71) $5(1-6k)+10(k-8)$

72) $-7(4x+3)-10(10x+10)$

73) $(8n^2-3n)-(5+4n^2)$

74) $(7x^2-3)-(5x^2+6x)$

75) $(5p-6)+(1-p)$

76) $(3x^2-x)-(7-8x)$

77) $(2-4v^2)+(3v^2+2v)$

78) $(2b-8)+(b-7b^2)$

79) $(4-2k^2)+(8-2k^2)$

80) $(7a^2+7a)-(6a^2+4a)$

81) $(x^2-8)+(2x^2-7)$

82) $(3-7n^2)+(6n^2+3)$

Chapter 1 : Solving Linear Equations

1.1

Solving Linear Equations - One Step Equations

Objective: Solve one step linear equations by balancing using inverse operations

Solving linear equations is an important and fundamental skill in algebra. In algebra, we are often presented with a problem where the answer is known, but part of the problem is missing. The missing part of the problem is what we seek to find. An example of such a problem is shown below.

Example 41.

$$4x + 16 = -4$$

Notice the above problem has a missing part, or unknown, that is marked by x. If we are given that the solution to this equation is -5, it could be plugged into the equation, replacing the x with -5. This is shown in Example 2.

Example 42.

$$\begin{aligned} 4(-5) + 16 &= -4 && \text{Multiply } 4(-5) \\ -20 + 16 &= -4 && \text{Add } -20 + 16 \\ -4 &= -4 && \text{True!} \end{aligned}$$

Now the equation comes out to a true statement! Notice also that if another number, for example, 3, was plugged in, we would not get a true statement as seen in Example 3.

Example 43.

$$\begin{aligned} 4(3) + 16 &= -4 && \text{Multiply } 4(3) \\ 12 + 16 &= -4 && \text{Add } 12 + 16 \\ 28 &\neq -4 && \text{False!} \end{aligned}$$

Due to the fact that this is not a true statement, this demonstates that 3 is not the solution. However, depending on the complexity of the problem, this “guess and check” method is not very efficient. Thus, we take a more algebraic approach to solving equations. Here we will focus on what are called “one-step equations” or equations that only require one step to solve. While these equations often seem very fundamental, it is important to master the pattern for solving these problems so we can solve more complex problems.

Addition Problems

To solve equations, the general rule is to do the opposite. For example, consider the following example.

Example 44.

$$\begin{array}{rl} x+7=-5 & \text{The 7 is added to the } x \\ \underline{\mathbf{-7 \quad -7}} & \text{Subtract 7 from both sides to get rid of it} \\ x=-12 & \text{Our solution!} \end{array}$$

Then we get our solution, $x = -12$. The same process is used in each of the following examples.

Example 45.

$4+x=8$	$7=x+9$	$5=8+x$
$\underline{\mathbf{-4 \quad -4}}$	$\underline{\mathbf{-9 \quad -9}}$	$\underline{\mathbf{-8 \; -8}}$
$x=4$	$-2=x$	$-3=x$

Table 1. Addition Examples

Subtraction Problems

In a subtraction problem, we get rid of negative numbers by adding them to both sides of the equation. For example, consider the following example.

Example 46.

$$\begin{array}{rl} x-5=4 & \text{The 5 is negative, or subtracted from } x \\ \underline{\mathbf{+5+5}} & \text{Add 5 to both sides} \\ x=9 & \text{Our Solution!} \end{array}$$

Then we get our solution $x = 9$. The same process is used in each of the following examples. Notice that each time we are getting rid of a negative number by adding.

Example 47.

$$\begin{array}{c}-6+x=-2\\ \underline{\mathbf{+6\qquad +6}}\\ x=4\end{array}\qquad\begin{array}{c}-10=x-7\\ \underline{\mathbf{+7\quad +7}}\\ -3=x\end{array}\qquad\begin{array}{c}5=-8+x\\ \underline{\mathbf{+8\ \ +8}}\\ 13=x\end{array}$$

Table 2. Subtraction Examples

Multiplication Problems

With a multiplication problem, we get rid of the number by dividing on both sides. For example consider the following example.

Example 48.

$$\begin{array}{rl}4x=20 & \text{Variable is multiplied by } 4\\ \overline{\mathbf{4}}\quad\overline{\mathbf{4}} & \text{Divide both sides by } 4\\ x=5 & \text{Our solution!}\end{array}$$

Then we get our solution $x=5$

With multiplication problems it is very important that care is taken with signs. If x is multiplied by a negative then we will divide by a negative. This is shown in example 9.

Example 49.

$$\begin{array}{rl}-5x=\ 30 & \text{Variable is multiplied by } -5\\ \overline{\mathbf{-5}}\quad\overline{\mathbf{-5}} & \text{Divide both sides by } -5\\ x=-6 & \text{Our Solution!}\end{array}$$

The same process is used in each of the following examples. Notice how negative and positive numbers are handled as each problem is solved.

Example 50.

$\dfrac{8x}{\mathbf{8}} = \dfrac{-24}{\mathbf{8}}$	$\dfrac{-4x}{\mathbf{-4}} = \dfrac{-20}{\mathbf{-4}}$	$\dfrac{42}{\mathbf{7}} = \dfrac{7x}{\mathbf{7}}$
$x = -3$	$x = 5$	$6 = x$

Table 3. Multiplication Examples

Division Problems:

In division problems, we get rid of the denominator by multiplying on both sides. For example consider our next example.

Example 51.

$$\begin{aligned} \frac{x}{5} &= -3 && \text{Variable is divided by 5} \\ (\mathbf{5})\frac{x}{5} &= -3(\mathbf{5}) && \text{Multiply both sides by 5} \\ x &= -15 && \text{Our Solution!} \end{aligned}$$

Then we get our solution $x = -15$. The same process is used in each of the following examples.

Example 52.

$\frac{x}{-7} = -2$	$\frac{x}{8} = 5$	$\frac{x}{-4} = 9$
$(\mathbf{-7})\frac{x}{-7} = -2(\mathbf{-7})$	$(\mathbf{8})\frac{x}{8} = 5(\mathbf{8})$	$(\mathbf{-4})\frac{x}{-4} = 9(\mathbf{-4})$
$x = 14$	$x = 40$	$x = -36$

Table 4. Division Examples

The process described above is fundamental to solving equations. once this process is mastered, the problems we will see have several more steps. These problems may seem more complex, but the process and patterns used will remain the same.

World View Note: The study of algebra originally was called the "Cossic Art" from the Latin, the study of "things" (which we now call variables).

1.1 Practice - One Step Equations

Solve each equation.

1) $v+9=16$

2) $14=b+3$

3) $x-11=-16$

4) $-14=x-18$

5) $30=a+20$

6) $-1+k=5$

7) $x-7=-26$

8) $-13+p=-19$

9) $13=n-5$

10) $22=16+m$

11) $340=-17x$

12) $4r=-28$

13) $-9=\frac{n}{12}$

14) $\frac{5}{9}=\frac{b}{9}$

15) $20v=-160$

16) $-20x=-80$

17) $340=20n$

18) $\frac{1}{2}=\frac{a}{8}$

19) $16x=320$

20) $\frac{k}{13}=-16$

21) $-16+n=-13$

22) $21=x+5$

23) $p-8=-21$

24) $m-4=-13$

25) $180=12x$

26) $3n=24$

27) $20b=-200$

28) $-17=\frac{x}{12}$

29) $\frac{r}{14}=\frac{5}{14}$

30) $n+8=10$

31) $-7=a+4$

32) $v-16=-30$

33) $10=x-4$

34) $-15=x-16$

35) $13a=-143$

36) $-8k=120$

37) $\frac{p}{20}=-12$

38) $-15=\frac{x}{9}$

39) $9+m=-7$

40) $-19=\frac{n}{20}$

1.2

Linear Equations - Two-Step Equations

Objective: Solve two-step equations by balancing and using inverse opperations.

After mastering the technique for solving equations that are simple one-step equations, we are ready to consider two-step equations. As we solve two-step equations, the important thing to remember is that everything works backwards! When working with one-step equations, we learned that in order to clear a "plus five" in the equation, we would subtract five from both sides. We learned that to clear "divided by seven" we multiply by seven on both sides. The same pattern applies to the order of operations. When solving for our variable x, we use order of operations backwards as well. This means we will add or subtract first, then multiply or divide second (then exponents, and finally any parentheses or grouping symbols, but that's another lesson). So to solve the equation in the first example,

Example 53.

$$4x - 20 = -8$$

We have two numbers on the same side as the x. We need to move the 4 and the 20 to the other side. We know to move the four we need to divide, and to move the twenty we will add twenty to both sides. If order of operations is done backwards, we will add or subtract first. Therefore we will add 20 to both sides first. Once we are done with that, we will divide both sides by 4. The steps are shown below.

$4x - 20 = -8$	Start by focusing on the subtract 20
$\underline{\mathbf{+20\ +20}}$	Add 20 to both sides
$4x \quad = 12$	Now we focus on the 4 multiplied by x
$\overline{\mathbf{4}} \quad \overline{\mathbf{4}}$	Divide both sides by 4
$x = 3$	Our Solution!

Notice in our next example when we replace the x with 3 we get a true statement.

$4(3) - 20 = -8$	Multiply $4(3)$
$12 - 20 = -8$	Subtract $12 - 20$
$-8 = -8$	True!

The same process is used to solve any two-step equations. Add or subtract first, then multiply or divide. Consider our next example and notice how the same process is applied.

Example 54.

$$\begin{array}{rl}
5x+7=\ \ 7 & \text{Start by focusing on the plus 7}\\
\underline{\mathbf{-7}\ \ \mathbf{-7}} & \text{Subtract 7 from both sides}\\
5x\ \ \ =\ 0 & \text{Now focus on the multiplication by 5}\\
\overline{\mathbf{5}}\ \ \ \ \overline{\mathbf{5}} & \text{Divide both sides by 5}\\
x=\ \ 0 & \text{Our Solution!}
\end{array}$$

Notice the seven subtracted out completely! Many students get stuck on this point, do not forget that we have a number for “nothing left” and that number is zero. With this in mind the process is almost identical to our first example.

A common error students make with two-step equations is with negative signs. Remember the sign always stays with the number. Consider the following example.

Example 55.

$$\begin{array}{rl}
4-2x=10 & \text{Start by focusing on the positive 4}\\
\underline{\mathbf{-4}\ \ \ \ \ \ \ \ \mathbf{-4}} & \text{Subtract 4 from both sides}\\
-2x\ =6 & \text{Negative (subtraction) stays on the } 2x\\
\overline{\mathbf{-2}}\ \ \overline{\mathbf{-2}} & \text{Divide by } -2\\
x=-3 & \text{Our Solution!}
\end{array}$$

The same is true even if there is no coefficient in front of the variable. Consider the next example.

Example 56.

$$\begin{array}{rl}
8-x=2 & \text{Start by focusing on the positive 8}\\
\underline{\mathbf{-8}\ \ \ \ \ \mathbf{-8}} & \text{Subtract 8 from both sides}\\
-x=-6 & \text{Negative (subtraction) stays on the } x\\
-1x=-6 & \text{Remember, no number in front of variable means 1}
\end{array}$$

$$\begin{array}{rl}\overline{\mathbf{-1}}\quad\overline{\mathbf{-1}} & \text{Divide both sides by } -1\\ x=6 & \text{Our Solution!}\end{array}$$

Solving two-step equations is a very important skill to master, as we study algebra. The first step is to add or subtract, the second is to multiply or divide. This pattern is seen in each of the following examples.

Example 57.

$$\begin{array}{r}-3x+7=-8\\ \underline{\mathbf{-7}\ \ \mathbf{-7}}\\ -3x=-15\\ \overline{\mathbf{-3}}\quad\overline{\mathbf{-3}}\\ x=5\end{array}\qquad\qquad \begin{array}{r}-2+9x=7\\ \underline{\mathbf{+2}\qquad\mathbf{+2}}\\ 9x=9\\ \overline{\mathbf{9}}\quad\overline{\mathbf{9}}\\ x=1\end{array}\qquad\qquad \begin{array}{r}8=2x+10\\ \underline{\mathbf{-10}\qquad\mathbf{-10}}\\ -2=2x\\ \overline{\mathbf{2}}\quad\overline{\mathbf{2}}\\ -1=x\end{array}$$

$$\begin{array}{r}7-5x=17\\ \underline{\mathbf{-7}\qquad\mathbf{-7}}\\ -5x=10\\ \overline{\mathbf{-5}}\ \overline{\mathbf{-5}}\\ x=-2\end{array}\qquad\qquad \begin{array}{r}-5-3x=-5\\ \underline{\mathbf{+5}\qquad\mathbf{+5}}\\ -3x=0\\ \overline{\mathbf{-3}}\ \overline{\mathbf{-3}}\\ x=0\end{array}\qquad\qquad \begin{array}{c}-3=\frac{x}{5}-4\\ \underline{\mathbf{+4}\quad\mathbf{+4}}\\ \mathbf{(5)}(1)=\frac{x}{5}\mathbf{(5)}\\ 5=x\end{array}$$

Table 5. Two-Step Equation Examples

As problems in algebra become more complex the process covered here will remain the same. In fact, as we solve problems like those in the next example, each one of them will have several steps to solve, but the last two steps are a two-step equation like we are solving here. This is why it is very important to master two-step equations now!

Example 58.

$$3x^2+4-x+6\qquad \frac{1}{x-8}+\frac{1}{x}=\frac{1}{3}\qquad \sqrt{5x-5}+1=x\qquad \log_5(2x-4)=1$$

World View Note: Persian mathematician Omar Khayyam would solve algebraic problems geometrically by intersecting graphs rather than solving them algebraically.

1.2 Practice - Two-Step Problems

Solve each equation.

1) $5+\frac{n}{4}=4$

2) $-2=-2m+12$

3) $102=-7r+4$

4) $27=21-3x$

5) $-8n+3=-77$

6) $-4-b=8$

7) $0=-6v$

8) $-2+\frac{x}{2}=4$

9) $-8=\frac{x}{5}-6$

10) $-5=\frac{a}{4}-1$

11) $0=-7+\frac{k}{2}$

12) $-6=15+3p$

13) $-12+3x=0$

14) $-5m+2=27$

15) $24=2n-8$

16) $-37=8+3x$

17) $2=-12+2r$

18) $-8+\frac{n}{12}=-7$

19) $\frac{b}{3}+7=10$

20) $\frac{x}{1}-8=-8$

21) $152=8n+64$

22) $-11=-8+\frac{v}{2}$

23) $-16=8a+64$

24) $-2x-3=-29$

25) $56+8k=64$

26) $-4-3n=-16$

27) $-2x+4=22$

28) $67=5m-8$

29) $-20=4p+4$

30) $9=8+\frac{x}{6}$

31) $-5=3+\frac{n}{2}$

32) $\frac{m}{4}-1=-2$

33) $\frac{r}{8}-6=-5$

34) $-80=4x-28$

35) $-40=4n-32$

36) $33=3b+3$

37) $87=3-7v$

38) $3x-3=-3$

39) $-x+1=-11$

40) $4+\frac{a}{3}=1$

1.3

Solving Linear Equations - General Equations

Objective: Solve general linear equations with variables on both sides.

Often as we are solving linear equations we will need to do some work to set them up into a form we are familiar with solving. This section will focus on manipulating an equation we are asked to solve in such a way that we can use our pattern for solving two-step equations to ultimately arrive at the solution.

One such issue that needs to be addressed is parenthesis. Often the parenthesis can get in the way of solving an otherwise easy problem. As you might expect we can get rid of the unwanted parenthesis by using the distributive property. This is shown in the following example. Notice the first step is distributing, then it is solved like any other two-step equation.

Example 59.

$$\begin{array}{rl}
4(2x-6)=16 & \text{Distribute 4 through parenthesis} \\
8x-24=16 & \text{Focus on the subtraction first} \\
\underline{+\mathbf{24}+\mathbf{24}} & \text{Add 24 to both sides} \\
8x=40 & \text{Now focus on the multiply by 8} \\
\overline{\mathbf{8}}\ \ \overline{\mathbf{8}} & \text{Divide both sides by 8} \\
x=5 & \text{Our Solution!}
\end{array}$$

Often after we distribute there will be some like terms on one side of the equation. Example 2 shows distributing to clear the parenthesis and then combining like terms next. Notice we only combine like terms on the same side of the equation. Once we have done this, our next example solves just like any other two-step equation.

Example 60.

$$\begin{array}{rl}
3(2x-4)+9=15 & \text{Distribute the 3 through the parenthesis} \\
6x-12+9=15 & \text{Combine like terms, } -12+9 \\
6x-3=15 & \text{Focus on the subtraction first} \\
\underline{+3\ \ +3} & \text{Add 3 to both sides} \\
6x=18 & \text{Now focus on multiply by 6}
\end{array}$$

$$\begin{array}{rl} \overline{6} \quad \overline{6} & \text{Divide both sides by 6} \\ x=3 & \text{Our Solution} \end{array}$$

A second type of problem that becomes a two-step equation after a bit of work is one where we see the variable on both sides. This is shown in the following example.

Example 61.

$$4x-6=2x+10$$

Notice here the x is on both the left and right sides of the equation. This can make it difficult to decide which side to work with. We fix this by moving one of the terms with x to the other side, much like we moved a constant term. It doesn't matter which term gets moved, $4x$ or $2x$, however, it would be the author's suggestion to move the smaller term (to avoid negative coefficients). For this reason we begin this problem by clearing the positive $2x$ by subtracting $2x$ from both sides.

$$\begin{array}{rl} 4x-6=2x+10 & \text{Notice the variable on both sides} \\ \underline{-\mathbf{2x} \qquad -\mathbf{2x}} & \text{Subtract } 2x \text{ from both sides} \\ 2x-6=10 & \text{Focus on the subtraction first} \\ \underline{+\mathbf{6}+\mathbf{6}} & \text{Add 6 to both sides} \\ 2x=16 & \text{Focus on the multiplication by 2} \\ \overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \text{Divide both sides by 2} \\ x=8 & \text{Our Solution!} \end{array}$$

The previous example shows the check on this solution. Here the solution is plugged into the x on both the left and right sides before simplifying.

Example 62.

$$\begin{array}{rl} 4(8)-6=2(8)+10 & \text{Multiply } 4(8) \text{ and } 2(8) \text{ first} \\ 32-6=16+10 & \text{Add and Subtract} \\ 26=26 & \text{True!} \end{array}$$

The next example illustrates the same process with negative coefficients. Notice first the smaller term with the variable is moved to the other side, this time by adding because the coefficient is negative.

Example 63.

$$
\begin{array}{rl}
-3x+9=6x-27 & \text{Notice the variable on both sides, } -3x \text{ is smaller} \\
\underline{+\mathbf{3x} \quad\quad +\mathbf{3x}} & \text{Add } 3x \text{ to both sides} \\
9=9x-27 & \text{Focus on the subtraction by } 27 \\
\underline{+\mathbf{27} \quad\quad +\mathbf{27}} & \text{Add } 27 \text{ to both sides} \\
36=9x & \text{Focus on the mutiplication by } 9 \\
\overline{\mathbf{9}} \quad \overline{\mathbf{9}} & \text{Divide both sides by } 9 \\
4=x & \text{Our Solution}
\end{array}
$$

Linear equations can become particularly intersting when the two processes are combined. In the following problems we have parenthesis and the variable on both sides. Notice in each of the following examples we distribute, then combine like terms, then move the variable to one side of the equation.

Example 64.

$$
\begin{array}{rl}
2(x-5)+3x=x+18 & \text{Distribute the 2 through parenthesis} \\
2x-10+3x=x+18 & \text{Combine like terms } 2x+3x \\
5x-10=x+18 & \text{Notice the variable is on both sides} \\
\underline{-\boldsymbol{x} \quad\quad -\boldsymbol{x}} & \text{Subtract } x \text{ from both sides} \\
4x-10=18 & \text{Focus on the subtraction of } 10 \\
\underline{+\mathbf{10}+\mathbf{10}} & \text{Add } 10 \text{ to both sides} \\
4x=28 & \text{Focus on multiplication by } 4 \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by } 4 \\
x=7 & \text{Our Solution}
\end{array}
$$

Sometimes we may have to distribute more than once to clear several parenthesis. Remember to combine like terms after you distribute!

Example 65.

$$
\begin{array}{rl}
3(4x-5)-4(2x+1)=5 & \text{Distribute 3 and } -4 \text{ through parenthesis} \\
12x-15-8x-4=5 & \text{Combine like terms } 12x-8x \text{ and } -15-4 \\
4x-19 \quad =5 & \text{Focus on subtraction of } 19 \\
\underline{+\mathbf{19}+\mathbf{19}} & \text{Add } 19 \text{ to both sides} \\
4x=24 & \text{Focus on multiplication by } 4
\end{array}
$$

$$\begin{array}{rl} \underline{\mathbf{4}}\quad \underline{\mathbf{4}} & \text{Divide both sides by 4} \\ x=6 & \text{Our Solution} \end{array}$$

This leads to a 5-step process to solve any linear equation. While all five steps aren't always needed, this can serve as a guide to solving equations.

1. Distribute through any parentheses.

2. Combine like terms on each side of the equation.

3. Get the variables on one side by adding or subtracting

4. Solve the remaining 2-step equation (add or subtract then multiply or divide)

5. Check your answer by plugging it back in for x to find a true statement.

The order of these steps is very important.

World View Note: The Chinese developed a method for solving equations that involved finding each digit one at a time about 2000 years ago!

We can see each of the above five steps worked through our next example.

Example 66.

$$\begin{array}{rl} 4(2x-6)+9=3(x-7)+8x & \text{Distribute 4 and 3 through parenthesis} \\ 8x-24+9=3x-21+8x & \text{Combine like terms } -24+9 \text{ and } 3x+8x \\ 8x-15=11x-21 & \text{Notice the variable is on both sides} \\ \underline{-\mathbf{8x}\qquad -\mathbf{8x}} & \text{Subtract } 8x \text{ from both sides} \\ -15=3x-21 & \text{Focus on subtraction of 21} \\ \underline{+\mathbf{21}\qquad +\mathbf{21}} & \text{Add 21 to both sides} \\ 6=3x & \text{Focus on multiplication by 3} \\ \underline{\mathbf{3}}\quad \underline{\mathbf{3}} & \text{Divide both sides by 3} \\ 2=x & \text{Our Solution} \end{array}$$

Check:

$$\begin{array}{rl} 4[2(2)-6]+9=3[(2)-7]+8(2) & \text{Plug 2 in for each } x\text{. Multiply inside parenthesis} \\ 4[4-6]+9=3[-5]+8(2) & \text{Finish parentesis on left, multiply on right} \end{array}$$

$$\begin{aligned} 4[-2]+9 &= -15+8(2) && \text{Finish multiplication on both sides} \\ -8+9 &= -15+16 && \text{Add} \\ 1 &= 1 && \text{True!} \end{aligned}$$

When we check our solution of $x=2$ we found a true statement, $1=1$. Therefore, we know our solution $x=2$ is the correct solution for the problem.

There are two special cases that can come up as we are solving these linear equations. The first is illustrated in the next two examples. Notice we start by distributing and moving the variables all to the same side.

Example 67.

$$\begin{aligned} 3(2x-5) &= 6x-15 && \text{Distribute 3 through parenthesis} \\ 6x-15 &= 6x-15 && \text{Notice the variable on both sides} \\ \underline{\boldsymbol{-6x} \quad} & \underline{\quad \boldsymbol{-6x}} && \text{Subtract } 6x \text{ from both sides} \\ -15 &= -15 && \text{Variable is gone! True!} \end{aligned}$$

Here the variable subtracted out completely! We are left with a true statement, $-15=-15$. If the variables subtract out completely and we are left with a true statement, this indicates that the equation is always true, no matter what x is. Thus, for our solution we say **all real numbers** or $\mathbb{R}$.

Example 68.

$$\begin{aligned} 2(3x-5)-4x &= 2x+7 && \text{Distribute 2 through parenthesis} \\ 6x-10-4x &= 2x+7 && \text{Combine like terms } 6x-4x \\ 2x-10 &= 2x+7 && \text{Notice the variable is on both sides} \\ \underline{\boldsymbol{-2x} \quad} & \underline{\quad \boldsymbol{-2x}} && \text{Subtract } 2x \text{ from both sides} \\ -10 &\neq 7 && \text{Variable is gone! False!} \end{aligned}$$

Again, the variable subtracted out completely! However, this time we are left with a false statement, this indicates that the equation is never true, no matter what x is. Thus, for our solution we say **no solution** or $\varnothing$.

1.3 Practice - General Linear Equations

Solve each equation.

1) $2-(-3a-8)=1$

2) $2(-3n+8)=-20$

3) $-5(-4+2v)=-50$

4) $2-8(-4+3x)=34$

5) $66=6(6+5x)$

6) $32=2-5(-4n+6)$

7) $0=-8(p-5)$

8) $-55=8+7(k-5)$

9) $-2+2(8x-7)=-16$

10) $-(3-5n)=12$

11) $-21x+12=-6-3x$

12) $-3n-27=-27-3n$

13) $-1-7m=-8m+7$

14) $56p-48=6p+2$

15) $1-12r=29-8r$

16) $4+3x=-12x+4$

17) $20-7b=-12b+30$

18) $-16n+12=39-7n$

19) $-32-24v=34-2v$

20) $17-2x=35-8x$

21) $-2-5(2-4m)=33+5m$

22) $-25-7x=6(2x-1)$

23) $-4n+11=2(1-8n)+3n$

24) $-7(1+b)=-5-5b$

25) $-6v-29=-4v-5(v+1)$

26) $-8(8r-2)=3r+16$

27) $2(4x-4)=-20-4x$

28) $-8n-19=-2(8n-3)+3n$

29) $-a-5(8a-1)=39-7a$

30) $-4+4k=4(8k-8)$

31) $-57=-(-p+1)+2(6+8p)$

32) $16=-5(1-6x)+3(6x+7)$

33) $-2(m-2)+7(m-8)=-67$

34) $7=4(n-7)+5(7n+7)$

35) $50=8(7+7r)-(4r+6)$

36) $-8(6+6x)+4(-3+6x)=-12$

37) $-8(n-7)+3(3n-3)=41$

38) $-76=5(1+3b)+3(3b-3)$

39) $-61=-5(5r-4)+4(3r-4)$

40) $-6(x-8)-4(x-2)=-4$

41) $-2(8n-4)=8(1-n)$

42) $-4(1+a)=2a-8(5+3a)$

43) $-3(-7v+3)+8v=5v-4(1-6v)$

44) $-6(x-3)+5=-2-5(x-5)$

45) $-7(x-2)=-4-6(x-1)$

46) $-(n+8)+n=-8n+2(4n-4)$

47) $-6(8k+4)=-8(6k+3)-2$

48) $-5(x+7)=4(-8x-2)$

49) $-2(1-7p)=8(p-7)$

50) $8(-8n+4)=4(-7n+8)$

1.4

Solving Linear Equations - Fractions

Objective: Solve linear equations with rational coefficients by multiplying by the least common denominator to clear the fractions.

Often when solving linear equations we will need to work with an equation with fraction coefficients. We can solve these problems as we have in the past. This is demonstrated in our next example.

Example 69.

$$\begin{array}{rl} \frac{3}{4}x - \frac{7}{2} = \frac{5}{6} & \text{Focus on subtraction} \\ \underline{+\mathbf{\frac{7}{2}} \quad +\mathbf{\frac{7}{2}}} & \text{Add } \frac{7}{2} \text{ to both sides} \end{array}$$

Notice we will need to get a common denominator to add $\frac{5}{6} + \frac{7}{2}$. Notice we have a common denominator of 6. So we build up the denominator, $\frac{7}{2}\left(\frac{3}{3}\right) = \frac{21}{6}$, and we can now add the fractions:

$$\begin{array}{rl} \frac{3}{4}x - \frac{21}{6} = \frac{5}{6} & \text{Same problem, with common denominator 6} \\ \underline{+\mathbf{\frac{21}{6}} + \mathbf{\frac{21}{6}}} & \text{Add } \frac{21}{6} \text{ to both sides} \\ \frac{3}{4}x = \frac{26}{6} & \text{Reduce } \frac{26}{6} \text{ to } \frac{13}{3} \\ \frac{3}{4}x = \frac{13}{3} & \text{Focus on multiplication by } \frac{3}{4} \end{array}$$

We can get rid of $\frac{3}{4}$ by dividing both sides by $\frac{3}{4}$. Dividing by a fraction is the same as multiplying by the reciprocal, so we will multiply both sides by $\frac{4}{3}$.

$$\begin{array}{rl} \left(\mathbf{\frac{4}{3}}\right)\frac{3}{4}x = \frac{13}{3}\left(\mathbf{\frac{4}{3}}\right) & \text{Multiply by reciprocal} \\ x = \frac{52}{9} & \text{Our solution!} \end{array}$$

While this process does help us arrive at the correct solution, the fractions can make the process quite difficult. This is why we have an alternate method for dealing with fractions - clearing fractions. Clearing fractions is nice as it gets rid of the fractions for the majority of the problem. We can easily clear the fractions by finding the LCD and multiplying each term by the LCD. This is shown in the next example, the same problem as our first example, but this time we will solve by clearing fractions.

Example 70.

$$\frac{3}{4}x - \frac{7}{2} = \frac{5}{6} \quad \text{LCD} = 12\text{, multiply each term by } 12$$

$$\frac{(\mathbf{12})3}{4}x - \frac{(\mathbf{12})7}{2} = \frac{(\mathbf{12})5}{6} \quad \text{Reduce each 12 with denominators}$$

$$\begin{array}{rl}
(\mathbf{3})3x - (\mathbf{6})7 = (\mathbf{2})5 & \text{Multiply out each term} \\
9x - 42 = 10 & \text{Focus on subtraction by } 42 \\
\underline{+\mathbf{42} + \mathbf{42}} & \text{Add } 42 \text{ to both sides} \\
9x = 52 & \text{Focus on multiplication by } 9 \\
\overline{\mathbf{9}} \quad \overline{\mathbf{9}} & \text{Divide both sides by } 9 \\
x = \frac{52}{9} & \text{Our Solution}
\end{array}$$

The next example illustrates this as well. Notice the 2 isn't a fraction in the origional equation, but to solve it we put the 2 over 1 to make it a fraction.

Example 71.

$$\frac{2}{3}x - 2 = \frac{3}{2}x + \frac{1}{6} \quad \text{LCD} = 6\text{, multiply each term by } 6$$

$$\frac{(\mathbf{6})2}{3}x - \frac{(\mathbf{6})2}{1} = \frac{(\mathbf{6})3}{2}x + \frac{(\mathbf{6})1}{6} \quad \text{Reduce 6 with each denominator}$$

$$\begin{array}{rl}
(\mathbf{2})2x - (\mathbf{6})2 = (\mathbf{3})3x + (\mathbf{1})1 & \text{Multiply out each term} \\
4x - 12 = 9x + 1 & \text{Notice variable on both sides} \\
\underline{-\mathbf{4x} \qquad -\mathbf{4x}} & \text{Subtract } 4x \text{ from both sides} \\
-12 = 5x + 1 & \text{Focus on addition of } 1 \\
\underline{-\mathbf{1} \qquad -\mathbf{1}} & \text{Subtract 1 from both sides} \\
-13 = 5x & \text{Focus on multiplication of } 5 \\
\overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by } 5 \\
-\frac{13}{5} = x & \text{Our Solution}
\end{array}$$

We can use this same process if there are parenthesis in the problem. We will first distribute the coefficient in front of the parenthesis, then clear the fractions. This is seen in the following example.

Example 72.

$$\begin{array}{rl}
\dfrac{3}{2}\left(\dfrac{5}{9}x+\dfrac{4}{27}\right)=3 & \text{Distribute } \dfrac{3}{2} \text{ through parenthesis, reducing if possible} \\
\dfrac{5}{6}x+\dfrac{2}{9}=3 & \text{LCD} = 18\text{, multiply each term by } 18 \\
\dfrac{\mathbf{(18)}5}{6}x+\dfrac{\mathbf{(18)}2}{9}=\dfrac{\mathbf{(18)}3}{9} & \text{Reduce 18 with each denominator} \\
\mathbf{(3)}5x+\mathbf{(2)}2=\mathbf{(18)}3 & \text{Multiply out each term} \\
15x+4=54 & \text{Focus on addition of 4} \\
\underline{-\mathbf{4}\quad-\mathbf{4}} & \text{Subtract 4 from both sides} \\
15x=50 & \text{Focus on multiplication by 15} \\
.\ \overline{\mathbf{15}}\quad\overline{\mathbf{15}} & \text{Divide both sides by 15. Reduce on right side.} \\
x=\dfrac{10}{3} & \text{Our Solution}
\end{array}$$

While the problem can take many different forms, the pattern to clear the fraction is the same, after distributing through any parentheses we multiply each term by the LCD and reduce. This will give us a problem with no fractions that is much easier to solve. The following example again illustrates this process.

Example 73.

$$\begin{array}{rl}
\dfrac{3}{4}x-\dfrac{1}{2}=\dfrac{1}{3}(\dfrac{3}{4}x+6)-\dfrac{7}{2} & \text{Distribute } \dfrac{1}{3}\text{, reduce if possible} \\
\dfrac{3}{4}x-\dfrac{1}{2}=\dfrac{1}{4}x+2-\dfrac{7}{2} & \text{LCD} = 4\text{, multiply each term by 4.} \\
\dfrac{\mathbf{(4)}3}{4}x-\dfrac{\mathbf{(4)}1}{2}=\dfrac{\mathbf{(4)}1}{4}x+\dfrac{\mathbf{(4)}2}{1}-\dfrac{\mathbf{(4)}7}{2} & \text{Reduce 4 with each denominator} \\
\mathbf{(1)}3x-\mathbf{(2)}1=\mathbf{(1)}1x+\mathbf{(4)}2-\mathbf{(2)}7 & \text{Multiply out each term} \\
3x-2=x+8-14 & \text{Combine like terms } 8-14 \\
3x-2=x-6 & \text{Notice variable on both sides} \\
\underline{-\boldsymbol{x}\qquad-\boldsymbol{x}} & \text{Subtract } x \text{ from both sides} \\
2x-2=-6 & \text{Focus on subtraction by 2} \\
\underline{+\mathbf{2}\quad+\mathbf{2}} & \text{Add 2 to both sides} \\
2x=-4 & \text{Focus on multiplication by 2} \\
\overline{\mathbf{2}}\qquad\overline{\mathbf{2}} & \text{Divide both sides by 2} \\
x=-2 & \text{Our Solution}
\end{array}$$

World View Note: The Egyptians were among the first to study fractions and linear equations. The most famous mathematical document from Ancient Egypt is the Rhind Papyrus where the unknown variable was called “heap”

1.4 Practice - Fractions

Solve each equation.

1) $\frac{3}{5}(1+p)=\frac{21}{20}$

2) $-\frac{1}{2}=\frac{3}{2}k+\frac{3}{2}$

3) $0=-\frac{5}{4}(x-\frac{6}{5})$

4) $\frac{3}{2}n-\frac{8}{3}=-\frac{29}{12}$

5) $\frac{3}{4}-\frac{5}{4}m=\frac{113}{24}$

6) $\frac{11}{4}+\frac{3}{4}r=\frac{163}{32}$

7) $\frac{635}{72}=-\frac{5}{2}(-\frac{11}{4}+x)$

8) $-\frac{16}{9}=-\frac{4}{3}(\frac{5}{3}+n)$

9) $2b+\frac{9}{5}=-\frac{11}{5}$

10) $\frac{3}{2}-\frac{7}{4}v=-\frac{9}{8}$

11) $\frac{3}{2}(\frac{7}{3}n+1)=\frac{3}{2}$

12) $\frac{41}{9}=\frac{5}{2}(x+\frac{2}{3})-\frac{1}{3}x$

13) $-a-\frac{5}{4}(-\frac{8}{3}a+1)=-\frac{19}{4}$

14) $\frac{1}{3}(-\frac{7}{4}k+1)-\frac{10}{3}k=-\frac{13}{8}$

15) $\frac{55}{6}=-\frac{5}{2}(\frac{3}{2}p-\frac{5}{3})$

16) $-\frac{1}{2}(\frac{2}{3}x-\frac{3}{4})-\frac{7}{2}x=-\frac{83}{24}$

17) $\frac{16}{9}=-\frac{4}{3}(-\frac{4}{3}n-\frac{4}{3})$

18) $\frac{2}{3}(m+\frac{9}{4})-\frac{10}{3}=-\frac{53}{18}$

19) $-\frac{5}{8}=\frac{5}{4}(r-\frac{3}{2})$

20) $\frac{1}{12}=\frac{4}{3}x+\frac{5}{3}(x-\frac{7}{4})$

21) $-\frac{11}{3}+\frac{3}{2}b=\frac{5}{2}(b-\frac{5}{3})$

22) $\frac{7}{6}-\frac{4}{3}n=-\frac{3}{2}n+2(n+\frac{3}{2})$

23) $-(-\frac{5}{2}x-\frac{3}{2})=-\frac{3}{2}+x$

24) $-\frac{149}{16}-\frac{11}{3}r=-\frac{7}{4}r-\frac{5}{4}(-\frac{4}{3}r+1)$

25) $\frac{45}{16}+\frac{3}{2}n=\frac{7}{4}n-\frac{19}{16}$

26) $-\frac{7}{2}(\frac{5}{3}a+\frac{1}{3})=\frac{11}{4}a+\frac{25}{8}$

27) $\frac{3}{2}(v+\frac{3}{2})=-\frac{7}{4}v-\frac{19}{6}$

28) $-\frac{8}{3}-\frac{1}{2}x=-\frac{4}{3}x-\frac{2}{3}(-\frac{13}{4}x+1)$

29) $\frac{47}{9}+\frac{3}{2}x=\frac{5}{3}(\frac{5}{2}x+1)$

30) $\frac{1}{3}n+\frac{29}{6}=2(\frac{4}{3}n+\frac{2}{3})$

1.5

Solving Linear Equations - Formulas

Objective: Solve linear formulas for a given variable.

Solving formulas is much like solving general linear equations. The only difference is we will have several varaibles in the problem and we will be attempting to solve for one specific variable. For example, we may have a formula such as $A = \pi r^2 + \pi$rs (formula for surface area of a right circular cone) and we may be interested in solving for the varaible s. This means we want to isolate the s so the equation has s on one side, and everything else on the other. So a solution might look like $s = \frac{A - \pi r^2}{\pi s}$. This second equation gives the same information as the first, they are algebraically equivalent, however, one is solved for the area, while the other is solved for s (slant height of the cone). In this section we will discuss how we can move from the first equation to the second.

When solving formulas for a variable we need to focus on the one varaible we are trying to solve for, all the others are treated just like numbers. This is shown in the following example. Two parallel problems are shown, the first is a normal one-step equation, the second is a formula that we are solving for x

Example 74.

$$\begin{array}{rrl} 3x = 12 & wx = z & \text{In both problems, } x \text{ is multiplied by something} \\ \overline{\mathbf{3}}\quad\overline{\mathbf{3}} & \overline{\boldsymbol{w}}\quad\overline{\boldsymbol{w}} & \text{To isolate the } x \text{ we divide by 3 or } w. \\ x = 4 & x = \dfrac{z}{w} & \text{Our Solution} \end{array}$$

We use the same process to solve $3x = 12$ for x as we use to solve $wx = z$ for x. Because we are solving for x we treat all the other variables the same way we would treat numbers. Thus, to get rid of the multiplication we divided by w. This same idea is seen in the following example.

Example 75.

$$\begin{array}{rll} m + n = p & \text{for } n & \text{Solving for } n, \text{ treat all other variables like numbers} \\ \underline{-\boldsymbol{m}\quad -\boldsymbol{m}} & & \text{Subtract } m \text{ from both sides} \\ n = p - m & & \text{Our Solution} \end{array}$$

As p and m are not like terms, they cannot be combined. For this reason we leave the expression as $p - m$. This same one-step process can be used with grouping symbols.

Example 76.

$$\begin{array}{rclll} \dfrac{a(x-y)}{\boldsymbol{(x-y)}} & = & \dfrac{b}{\boldsymbol{(x-y)}} & \text{for } a & \text{Solving for } a\text{, treat } (x-y) \text{ like } a \text{ number} \\ & & & & \text{Divide both sides by } (x-y) \\ & a = \dfrac{b}{x-y} & & & \text{Our Solution} \end{array}$$

Because $(x - y)$ is in parenthesis, if we are not searching for what is inside the parenthesis, we can keep them together as a group and divide by that group. However, if we are searching for what is inside the parenthesis, we will have to break up the parenthesis by distributing. The following example is the same formula, but this time we will solve for x.

Example 77.

$$\begin{array}{lll} a(x-y)=b & \text{for } x & \text{Solving for } x\text{, we need to distribute to clear parenthesis} \\ ax-ay=b & & \text{This is } a \text{ two} - \text{step equation, } ay \text{ is subtracted from our } x \text{ term} \\ \underline{\boldsymbol{+ay+ay}} & & \text{Add } ay \text{ to both sides} \\ ax=b+ay & & \text{The } x \text{ is multipied by } a \\ \overline{\boldsymbol{a}} \quad \overline{\boldsymbol{a}} & & \text{Divide both sides by } a \\ x=\dfrac{b+ay}{a} & & \text{Our Solution} \end{array}$$

Be very careful as we isolate x that we do not try and cancel the a on top and bottom of the fraction. This is not allowed if there is any adding or subtracting in the fraction. There is no reducing possible in this problem, so our final reduced answer remains $x = \frac{b+ay}{a}$. The next example is another two-step problem

Example 78.

$$\begin{array}{rll} y=mx+b & \text{for } m & \text{Solving for } m\text{, focus on addition first} \\ \underline{\boldsymbol{-b} \qquad \boldsymbol{-b}} & & \text{Subtract } b \text{ from both sides} \\ y-b=mx & & m \text{ is multipied by } x. \\ \overline{\boldsymbol{x}} \qquad \overline{\boldsymbol{x}} & & \text{Divide both sides by } x \\ \dfrac{y-b}{x}=m & & \text{Our Solution} \end{array}$$

It is important to note that we know we are done with the problem when the variable we are solving for is isolated or alone on one side of the equation and it does not appear anywhere on the other side of the equation.

The next example is also a two-step equation, it is the problem we started with at the beginning of the lesson.

Example 79.

$$\begin{array}{rll}
A = \pi r^2 + \pi r s & \text{for } s & \text{Solving for } s\text{, focus on what is added to the term with } s \\
\underline{-\boldsymbol{\pi r^2} \quad -\boldsymbol{\pi r^2}} & & \text{Subtract } \pi r^2 \text{ from both sides} \\
A - \pi r^2 = \pi r s & & s \text{ is multipied by } \pi r \\
\overline{\boldsymbol{\pi r}} \quad \overline{\boldsymbol{\pi r}} & & \text{Divide both sides by } \pi r \\
\dfrac{A - \pi r^2}{\pi r} = s & & \text{Our Solution}
\end{array}$$

Again, we cannot reduce the πr in the numerator and denominator because of the subtraction in the problem.

Formulas often have fractions in them and can be solved in much the same way we solved with fractions before. First identify the LCD and then multiply each term by the LCD. After we reduce there will be no more fractions in the problem so we can solve like any general equation from there.

Example 80.

$$\begin{array}{rll}
h = \dfrac{2m}{n} & \text{for } m & \text{To clear the fraction we use LCD} = n \\
(\boldsymbol{n})h = \dfrac{(\boldsymbol{n})2m}{n} & & \text{Multiply each term by } n \\
nh = 2m & & \text{Reduce } n \text{ with denominators} \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} & & \text{Divide both sides by } 2 \\
\dfrac{nh}{2} = m & & \text{Our Solution}
\end{array}$$

The same pattern can be seen when we have several fractions in our problem.

Example 81.

$$\begin{array}{rll}
\dfrac{a}{b} + \dfrac{c}{b} = e & \text{for } a & \text{To clear the fraction we use LCD} = b \\
\dfrac{(\boldsymbol{b})a}{b} + \dfrac{(\boldsymbol{b})c}{b} = e\,(\boldsymbol{b}) & & \text{Multiply each term by } b \\
a + c = eb & & \text{Reduce } b \text{ with denominators} \\
\underline{-\boldsymbol{c} \quad -\boldsymbol{c}} & & \text{Subtract } c \text{ from both sides} \\
a = eb - c & & \text{Our Solution}
\end{array}$$

Depending on the context of the problem we may find a formula that uses the same letter, one capital, one lowercase. These represent different values and we must be careful not to combine a capital variable with a lower case variable.

Example 82.

$$a = \frac{A}{2-b} \quad \text{for } b \quad \text{Use LCD } (2-b) \text{ as a group}$$

$$\mathbf{(2-b)}a=\frac{\mathbf{(2-b)}A}{2-b} \qquad \text{Multiply each term by } (2-b)$$

$$\begin{array}{rl}
(2-b)a=A & \text{reduce } (2-b) \text{ with denominator} \\
2a-\mathrm{ab}=A & \text{Distribute through parenthesis} \\
\underline{\mathbf{-2a} \qquad \mathbf{-2a}} & \text{Subtract } 2a \text{ from both sides} \\
-\mathrm{ab}=A-2a & \text{The } b \text{ is multipied by } -a \\
\overline{\mathbf{-a}} \qquad \overline{\mathbf{-a}} & \text{Divide both sides by } -a \\
b=\dfrac{A-2a}{-a} & \text{Our Solution}
\end{array}$$

Notice the A and a were not combined as like terms. This is because a formula will often use a capital letter and lower case letter to represent different variables. Often with formulas there is more than one way to solve for a variable. The next example solves the same problem in a slightly different manner. After clearing the denominator, we divide by a to move it to the other side, rather than distributing.

Example 83.

$$a=\frac{A}{2-b} \quad \text{for } b \quad \text{Use LCD} = (2-b) \text{ as } a \text{ group}$$

$$\begin{array}{rl}
\mathbf{(2-b)}a=\dfrac{\mathbf{(2-b)}A}{2-b} & \text{Multiply each term by } (2-b) \\
(2-b)a=A & \text{Reduce } (2-b) \text{ with denominator} \\
\overline{\qquad \mathbf{a}} \quad \overline{\mathbf{a}} & \text{Divide both sides by } a \\
2-b=\dfrac{A}{a} & \text{Focus on the positive 2} \\
\underline{\mathbf{-2} \qquad \mathbf{-2}} & \text{Subtract 2 from both sides} \\
-b=\dfrac{A}{a}-2 & \text{Still need to clear the negative} \\
\mathbf{(-1)}(-b)=\mathbf{(-1)}\dfrac{A}{a}-2\mathbf{(-1)} & \text{Multiply (or divide) each term by } -1 \\
b=-\dfrac{A}{a}+2 & \text{Our Solution}
\end{array}$$

Both answers to the last two examples are correct, they are just written in a different form because we solved them in different ways. This is very common with formulas, there may be more than one way to solve for a varaible, yet both are equivalent and correct.

World View Note: The father of algebra, Persian mathematician Muhammad ibn Musa Khwarizmi, introduced the fundamental idea of blancing by subtracting the same term to the other side of the equation. He called this process al-jabr which later became the world algebra.

1.5 Practice - Formulas

Solve each of the following equations for the indicated variable.

1) ab = c for b

2) g $= \frac{h}{i}$ for h

3) $\frac{f}{g}x = b$ for x

4) $p = \frac{3y}{q}$ for y

5) $3x = \frac{a}{b}$ for x

6) $\frac{\text{ym}}{b} = \frac{c}{d}$ for y

7) E = mc^2 for m

8) DS = ds for D

9) V $= \frac{4}{3}\pi r^3$ for π

10) E $= \frac{\text{mv}^2}{2}$ for m

11) a + c = b for c

12) $x - f = g$ for x

13) c $= \frac{4y}{m+n}$ for y

14) $\frac{\text{rs}}{a-3} = k$ for r

15) V $= \frac{\pi \text{Dn}}{12}$ for D

16) F = k$(R - L)$ for k

17) $P = n(p - c)$ for n

18) $S = L + 2B$ for L

19) $T = \frac{D-d}{L}$ for D

20) $I = \frac{E_a - E_q}{R}$ for E_a

21) L = $L_o(1 + \text{at})$ for L_o

22) ax + b = c for x

23) 2m + p = 4m + q for m

24) q = 6$(L - p)$ for L

25) $\frac{k-m}{r} = q$ for k

26) $R = \text{aT} + b$ for T

27) $h = \text{vt} - 16t^2$ for v

28) $S = \pi\text{rh} + \pi r^2$ for h

29) $Q_1 = P(Q_2 - Q_1)$ for Q_2

30) L = $\pi(r_1 + r_2) + 2d$ for r_1

31) $R = \frac{\text{kA}(T_1 + T_2)}{d}$ for T_1

32) $P = \frac{V_1(V_2 - V_1)}{g}$ for V_2

33) ax + b = c for a

34) rt = d for r

35) lwh = V for w

36) V $= \frac{\pi r^2 h}{3}$ for h

37) $\frac{1}{a} + b = \frac{c}{a}$ for a

38) $\frac{1}{a} + b = \frac{c}{a}$ for b

39) $\text{at} - \text{bw} = s$ for t

40) $\text{at} - \text{bw} = s$ for w

41) ax + bx = c for a

42) x + 5y = 3 for x

43) x + 5y = 3 for y

44) 3x + 2y = 7 for x

45) 3x + 2y = 7 for y

46) $5a - 7b = 4$ for a

47) $5a - 7b = 4$ for b

48) $4x - 5y = 8$ for x

49) $4x - 5y = 8$ for y

50) $C = \frac{5}{9}(F - 32)$ for F

1.6

Solving Linear Equations - Absolute Value

Objective: Solve linear absolute value equations.

When solving equations with absolute value we can end up with more than one possible answer. This is because what is in the absolute value can be either negative or positive and we must account for both possibilities when solving equations. This is illustrated in the following example.

Example 84.

$$\begin{array}{rl} |x| = 7 & \text{Absolute value can be positive or negative} \\ x = 7 \text{ or } x = -7 & \text{Our Solution} \end{array}$$

Notice that we have considered two possibilities, both the positive and negative. Either way, the absolute value of our number will be positive 7.

World View Note: The first set of rules for working with negatives came from 7th century India. However, in 1758, almost a thousand years later, British mathematician Francis Maseres claimed that negatives "Darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple."

When we have absolute values in our problem it is important to first isolate the absolute value, then remove the absolute value by considering both the positive and negative solutions. Notice in the next two examples, all the numbers outside of the absolute value are moved to the other side first before we remove the absolute value bars and consider both positive and negative solutions.

Example 85.

$$\begin{array}{rl} 5 + |x| = 8 & \text{Notice absolute value is not alone} \\ \underline{-5 \qquad -5} & \text{Subtract 5 from both sides} \\ |x| = 3 & \text{Absolute value can be positive or negative} \\ x = 3 \text{ or } x = -3 & \text{Our Solution} \end{array}$$

Example 86.

$$\begin{array}{rl} -4|x| = -20 & \text{Notice absolute value is not alone} \\ \overline{-4} \qquad \overline{-4} & \text{Divide both sides by } -4 \end{array}$$

$$
\begin{aligned}
|x| &= 5 && \text{Absolute value can be positive or negative}\\
x = 5 \text{ or } x &= -5 && \text{Our Solution}
\end{aligned}
$$

Notice we never combine what is inside the absolute value with what is outside the absolute value. This is very important as it will often change the final result to an incorrect solution. The next example requires two steps to isolate the absolute value. The idea is the same as a two-step equation, add or subtract, then multiply or divide.

Example 87.

$$
\begin{aligned}
5|x| - 4 &= 26 && \text{Notice the absolute value is not alone}\\
\underline{+\mathbf{4}\ \ +\mathbf{4}} & && \text{Add 4 to both sides}\\
5|x| &= 30 && \text{Absolute value still not alone}\\
\overline{\mathbf{5}}\quad \overline{\mathbf{5}} & && \text{Divide both sides by 5}\\
|x| &= 6 && \text{Absolute value can be positive or negative}\\
x = 6 \text{ or } x &= -6 && \text{Our Solution}
\end{aligned}
$$

Again we see the same process, get the absolute value alone first, then consider the positive and negative solutions. Often the absolute value will have more than just a variable in it. In this case we will have to solve the resulting equations when we consider the positive and negative possibilities. This is shown in the next example.

Example 88.

$$
\begin{aligned}
|2x - 1| &= 7 && \text{Absolute value can be positive or negative}\\
2x - 1 = 7 \text{ or } 2x - 1 &= -7 && \text{Two equations to solve}
\end{aligned}
$$

Now notice we have two equations to solve, each equation will give us a different solution. Both equations solve like any other two-step equation.

$$
\begin{array}{rcr}
2x - 1 = 7 & & 2x - 1 = -7\\
\underline{+\mathbf{1} + \mathbf{1}} & & \underline{+\mathbf{1}\ \ +\mathbf{1}}\\
2x = 8 & \text{or} & 2x = -6\\
\overline{\mathbf{2}}\quad \overline{\mathbf{2}} & & \overline{2}\quad \overline{2}\\
x = 4 & & x = -3
\end{array}
$$

Thus, from our previous example we have two solutions, $x=4$ or $x=-3$.

Again, it is important to remember that the absolute value must be alone first before we consider the positive and negative possibilities. This is illustrated in below.

Example 89.

$$2-4|2x+3|=-18$$

To get the absolute value alone we first need to get rid of the 2 by subtracting, then divide by -4. Notice we cannot combine the 2 and -4 becuase they are not like terms, the -4 has the absolute value connected to it. Also notice we do not distribute the -4 into the absolute value. This is because the numbers outside cannot be combined with the numbers inside the absolute value. Thus we get the absolute value alone in the following way:

$$\begin{array}{rl}
2-4|2x+3|=-18 & \text{Notice absolute value is not alone}\\
\underline{\mathbf{-2}\qquad\qquad\quad\mathbf{-2}} & \text{Subtract 2 from both sides}\\
-4|2x+3|=-20 & \text{Absolute value still not alone}\\
\overline{\mathbf{-4}}\qquad\quad\overline{\mathbf{-4}} & \text{Divide both sides by } -4\\
|2x+3|=5 & \text{Absoloute value can be positive or negative}\\
2x+3=5 \text{ or } 2x+3=-5 & \text{Two equations to solve}
\end{array}$$

Now we just solve these two remaining equations to find our solutions.

$$\begin{array}{rcr}
2x+3=5 & & 2x+3=-5\\
\underline{\mathbf{-3}\ \mathbf{-3}} & & \underline{\mathbf{-3}\ \ \mathbf{-3}}\\
2x=2 & \text{or} & 2x=-8\\
\overline{\mathbf{2}}\ \ \overline{\mathbf{2}} & & \overline{\mathbf{2}}\ \ \ \overline{\mathbf{2}}\\
x=1 & & x=-4
\end{array}$$

We now have our two solutions, $x=1$ and $x=-4$.

As we are solving absolute value equations it is important to be aware of special cases. Remember the result of an absolute value must always be positive. Notice what happens in the next example.

Example 90.

$$\begin{array}{rl} 7+|2x-5|=4 & \text{Notice absolute value is not alone} \\ \underline{\mathbf{-7 \qquad\qquad -7}} & \text{Subtract 7 from both sides} \\ |2x-5|=-3 & \text{Result of absolute value is negative!} \end{array}$$

Notice the absolute value equals a negative number! This is impossible with absolute value. When this occurs we say there is **no solution** or $\varnothing$.

One other type of absolute value problem is when two absolute values are equal to eachother. We still will consider both the positive and negative result, the difference here will be that we will have to distribute a negative into the second absolute value for the negative possibility.

Example 91.

$$\begin{array}{cl} |2x-7|=|4x+6| & \text{Absolute value can be positive or negative} \\ 2x-7=4x+6 \quad \text{or} \quad 2x-7=-(4x+6) & \text{make second part of second equation negative} \end{array}$$

Notice the first equation is the positive possibility and has no significant difference other than the missing absolute value bars. The second equation considers the negative possibility. For this reason we have a negative in front of the expression which will be distributed through the equation on the first step of solving. So we solve both these equations as follows:

$$\begin{array}{ccc} \begin{array}{r} 2x-7=4x+6 \\ \underline{\mathbf{-2x \quad -2x}} \\ -7=2x+6 \\ \underline{\mathbf{-6 \qquad -6}} \\ -13=2x \\ \overline{\mathbf{2}} \quad \overline{\mathbf{2}} \\ \dfrac{-13}{2}=x \end{array} & \text{or} & \begin{array}{r} 2x-7=-(4x+6) \\ 2x-7=-4x-6 \\ \underline{\mathbf{+4x \qquad +4x}} \\ 6x-7=-6 \\ \underline{\mathbf{+7 \quad +7}} \\ 6x=1 \\ \overline{\mathbf{6}} \quad \overline{\mathbf{6}} \\ x=\dfrac{1}{6} \end{array} \end{array}$$

This gives us our two solutions, $x=\frac{-13}{2}$ or $x=\frac{1}{6}$.

1.6 Practice - Absolute Value Equations

Solve each equation.

1) $|x| = 8$

2) $|n| = 7$

3) $|b| = 1$

4) $|x| = 2$

5) $|5 + 8a| = 53$

6) $|9n + 8| = 46$

7) $|3k + 8| = 2$

8) $|3 - x| = 6$

9) $|9 + 7x| = 30$

10) $|5n + 7| = 23$

11) $|8 + 6m| = 50$

12) $|9p + 6| = 3$

13) $|6 - 2x| = 24$

14) $|3n - 2| = 7$

15) $-7|-3 - 3r| = -21$

16) $|2 + 2b| + 1 = 3$

17) $7|-7x - 3| = 21$

18) $\frac{|-4 - 3n|}{4} = 2$

19) $\frac{|-4b - 10|}{8} = 3$

20) $8|5p + 8| - 5 = 11$

21) $8|x + 7| - 3 = 5$

22) $3 - |6n + 7| = -40$

23) $5|3 + 7m| + 1 = 51$

24) $4|r + 7| + 3 = 59$

25) $3 + 5|8 - 2x| = 63$

26) $5 + 8|-10n - 2| = 101$

27) $|6b - 2| + 10 = 44$

28) $7|10v - 2| - 9 = 5$

29) $-7 + 8|-7x - 3| = 73$

30) $8|3 - 3n| - 5 = 91$

31) $|5x + 3| = |2x - 1|$

32) $|2 + 3x| = |4 - 2x|$

33) $|3x - 4| = |2x + 3|$

34) $|\frac{2x - 5}{3}| = |\frac{3x + 4}{2}|$

35) $|\frac{4x - 2}{5}| = |\frac{6x + 3}{2}|$

36) $|\frac{3x + 2}{2}| = |\frac{2x - 3}{3}|$

1.9

Solving Linear Equations - Age Problems

Objective: Solve age problems by creating and solving a linear equation.

An application of linear equations is what are called age problems. When we are solving age problems we generally will be comparing the age of two people both now and in the future (or past). Using the clues given in the problem we will be working to find their current age. There can be a lot of information in these problems and we can easily get lost in all the information. To help us organize and solve our problem we will fill out a three by three table for each problem. An example of the basic structure of the table is below

	Age Now	Change
Person 1		
Person 2		

Table 6. Structure of Age Table

Normally where we see "Person 1" and "Person 2" we will use the name of the person we are talking about. We will use this table to set up the following example.

Example 110.

Adam is 20 years younger than Brian. In two years Brian will be twice as old as Adam. How old are they now?

	Age Now	$+2$
Adam		
Brian		

We use Adam and Brian for our persons
We use $+2$ for change because the second phrase is two years in the future

	Age Now	$+2$
Adam	$x-20$	
Brain	x	

Consider the "Now" part, **Adam is 20 years youger than Brian**. We are given information about Adam, not Brian. So Brian is x now. To show Adam is 20 years younger we subtract 20, Adam is $x-20$.

	Age Now	$+2$
Adam	$x-20$	$x-20+2$
Brian	x	$x+2$

Now the $+2$ column is filled in. This is done by adding 2 to both Adam$'s$ and Brian$'s$ now column as shown in the table.

	Age Now	$+2$
Adam	$x-20$	$x-18$
Brian	x	$x+2$

Combine like terms in Adam$'s$ future age: $-20+2$
This table is now filled out and we are ready to try and solve.

$$B = 2A$$

Our equation comes from the future statement: **Brian will be twice as old as Adam.** This means the younger, Adam, needs to be multiplied by 2.

$$(x+2) = 2(x-18)$$

Replace B and A with the information in their future cells, Adam (A) is replaced with $x-18$ and Brian (B) is replaced with $(x+2)$ This is the equation to solve!

$$x+2 = 2x-36$$

Distribute through parenthesis

$$\underline{-\boldsymbol{x} \quad -\boldsymbol{x}}$$

Subtract x from both sides to get variable on one side

$$2 = x - 36$$

Need to clear the -36

$$\underline{+\mathbf{36} \quad +\mathbf{36}}$$

Add 36 to both sides

$$38 = x$$

Our solution for x

	Age now
Adam	$38-20=18$
Brian	38

The first column will help us answer the question.
Replace the $x's$ with 38 and simplify.
Adam is 18 and Brian is 38

Solving age problems can be summarized in the following five steps. These five steps are guidelines to help organize the problem we are trying to solve.

1. Fill in the now column. The person we know nothing about is x.
2. Fill in the future/past collumn by adding/subtracting the change to the now column.
3. Make an equation for the relationship in the future. This is independent of the table.
4. Replace variables in equation with information in future cells of table
5. Solve the equation for x, use the solution to answer the question

These five steps can be seen illustrated in the following example.

Example 111.

Carmen is 12 years older than David. Five years ago the sum of their ages was 28. How old are they now?

	Age Now	-5
Carmen		
David		

Five years ago is -5 in the change column.

	Age Now	-5
Carmen	$x+12$	
David	x	

Carmen is 12 years older than David. We don$'t$ know about David so he is x, Carmen then is $x+12$

	Age Now	-5
Carmen	$x+12$	$x+12-5$
David	x	$x-5$

Subtract 5 from now column to get the change

	Age Now	-5
Carmen	$x+12$	$x+7$
David	x	$x-5$

Simplify by combining like terms $12-5$
Our table is ready!

$C+D=28$	The sum of their ages will be 29. So we add C and D
$(x+7)+(x-5)=28$	Replace C and D with the change cells.
$x+7+x-5=28$	Remove parenthesis
$2x+2=28$	Combine like terms $x+x$ and $7-5$
$\underline{\mathbf{-2\ \ -2}}$	Subtract 2 from both sides
$2x=26$	Notice x is multiplied by 2
$\overline{\mathbf{2}}\ \ \overline{\mathbf{2}}$	Divide both sides by 2
$x=13$	Our solution for x

	Age Now
Caremen	$13+12=25$
David	13

Replace x with 13 to answer the question
Carmen is 25 and David is 13

Sometimes we are given the sum of their ages right now. These problems can be tricky. In this case we will write the sum above the now column and make the first person's age now x. The second person will then turn into the subtraction problem total $-x$. This is shown in the next example.

Example 112.

The sum of the ages of Nicole and Kristin is 32. In two years Nicole will be three times as old as Kristin. How old are they now?

	32 Age Now	$+2$
Nicole	x	
Kristen	$32-x$	

The change is $+2$ for two years in the future
The total is placed above Age Now
The first person is x. The second becomes $32-x$

	Age Now	$+2$
Nicole	x	$x+2$
Kristen	$32-x$	$32-x+2$

Add 2 to each cell fill in the change column

	Age Now	$+2$
Nicole	x	$x+2$
Kristen	$32-x$	$34-x$

Combine like terms $32+2$, our table is done!

$N=3K$	Nicole is three times as old as Kristin.
$(x+2)=3(34-x)$	Replace variables with information in change cells
$x+2=102-3x$	Distribute through parenthesis
$\underline{\mathbf{+3x\qquad\qquad +3x}}$	Add $3x$ to both sides so variable is only on one side

$$\begin{array}{rl} 4x+2=102 & \text{Solve the two} - \text{step equation} \\ \underline{\mathbf{-2 \quad -2}} & \text{Subtract 2 from both sides} \\ 4x=100 & \text{The variable is multiplied by 4} \\ \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by 4} \\ x=25 & \text{Our solution for } x \end{array}$$

	Age Now
Nicole	25
Kristen	$32-25=7$

Plug 25 in for x in the now column
Nicole is 25 and Kristin is 7

A slight variation on age problems is to ask not how old the people are, but rather ask how long until we have some relationship about their ages. In this case we alter our table slightly. In the change column because we don't know the time to add or subtract we will use a variable, t, and add or subtract this from the now column. This is shown in the next example.

Example 113.

Louis is 26 years old. Her daughter is 4 years old. In how many years will Louis be double her daughter's age?

	Age Now	$+t$
Louis	26	
Daughter	4	

As we are given their ages now, these numbers go into the table. The change is unknown, so we write $+t$ for the change

	Age Now	$+t$
Louis	26	$26+t$
Daughter	4	$4+t$

Fill in the change column by adding t to each person's age. Our table is now complete.

$$\begin{array}{rl} L=2D & \text{Louis will be double her daughter} \\ (26+t)=2(4+t) & \text{Replace variables with information in change cells} \\ 26+t=8+2t & \text{Distribute through parenthesis} \\ \underline{\mathbf{-t \quad\ -t}} & \text{Subtract } t \text{ from both sides} \\ 26=8+t & \text{Now we have an 8 added to the } t \\ \underline{\mathbf{-8-8}} & \text{Subtract 8 from both sides} \\ 18=t & \text{In 18 years she will be double her daughter's age} \end{array}$$

Age problems have several steps to them. However, if we take the time to work through each of the steps carefully, keeping the information organized, the problems can be solved quite nicely.

World View Note: The oldest man in the world was Shigechiyo Izumi from Japan who lived to be 120 years, 237 days. However, his exact age has been disputed.

1.9 Practice - Age Problems

1. A boy is 10 years older than his brother. In 4 years he will be twice as old as his brother. Find the present age of each.

2. A father is 4 times as old as his son. In 20 years the father will be twice as old as his son. Find the present age of each.

3. Pat is 20 years older than his son James. In two years Pat will be twice as old as James. How old are they now?

4. Diane is 23 years older than her daughter Amy. In 6 years Diane will be twice as old as Amy. How old are they now?

5. Fred is 4 years older than Barney. Five years ago the sum of their ages was 48. How old are they now?

6. John is four times as old as Martha. Five years ago the sum of their ages was 50. How old are they now?

7. Tim is 5 years older than JoAnn. Six years from now the sum of their ages will be 79. How old are they now?

8. Jack is twice as old as Lacy. In three years the sum of their ages will be 54. How old are they now?

9. The sum of the ages of John and Mary is 32. Four years ago, John was twice as old as Mary. Find the present age of each.

10. The sum of the ages of a father and son is 56. Four years ago the father was 3 times as old as the son. Find the present age of each.

11. The sum of the ages of a china plate and a glass plate is 16 years. Four years ago the china plate was three times the age of the glass plate. Find the present age of each plate.

12. The sum of the ages of a wood plaque and a bronze plaque is 20 years. Four

years ago, the bronze plaque was one-half the age of the wood plaque. Find the present age of each plaque.

13. A is now 34 years old, and B is 4 years old. In how many years will A be twice as old as B?

14. A man's age is 36 and that of his daughter is 3 years. In how many years will the man be 4 times as old as his daughter?

15. An Oriental rug is 52 years old and a Persian rug is 16 years old. How many years ago was the Oriental rug four times as old as the Persian Rug?

16. A log cabin quilt is 24 years old and a friendship quilt is 6 years old. In how may years will the log cabin quilt be three times as old as the friendship quilt?

17. The age of the older of two boys is twice that of the younger; 5 years ago it was three times that of the younger. Find the age of each.

18. A pitcher is 30 years old, and a vase is 22 years old. How many years ago was the pitcher twice as old as the vase?

19. Marge is twice as old as Consuelo. The sum of their ages seven years ago was 13. How old are they now?

20. The sum of Jason and Mandy's age is 35. Ten years ago Jason was double Mandy's age. How old are they now?

21. A silver coin is 28 years older than a bronze coin. In 6 years, the silver coin will be twice as old as the bronze coin. Find the present age of each coin.

22. A sofa is 12 years old and a table is 36 years old. In how many years will the table be twice as old as the sofa?

23. A limestone statue is 56 years older than a marble statue. In 12 years, the limestone will be three times as old as the marble statue. Find the present age of the statues.

24. A pewter bowl is 8 years old, and a silver bowl is 22 years old. In how many years will the silver bowl be twice the age of the pewter bowl?

25. Brandon is 9 years older than Ronda. In four years the sum of their ages will be 91. How old are they now?

26. A kerosene lamp is 95 years old, and an electric lamp is 55 years old. How many years ago was the kerosene lamp twice the age of the electric lamp?

27. A father is three times as old as his son, and his daughter is 3 years younger

than the son. If the sum of their ages 3 years ago was 63 years, find the present age of the father.

28. The sum of Clyde and Wendy's age is 64. In four years, Wendy will be three times as old as Clyde. How old are they now?

29. The sum of the ages of two ships is 12 years. Two years ago, the age of the older ship was three times the age of the newer ship. Find the present age of each ship.

30. Chelsea's age is double Daniel's age. Eight years ago the sum of their ages was 32. How old are they now?

31. Ann is eighteen years older than her son. One year ago, she was three times as old as her son. How old are they now?

32. The sum of the ages of Kristen and Ben is 32. Four years ago Kristen was twice as old as Ben. How old are they both now?

33. A mosaic is 74 years older than the engraving. Thirty years ago, the mosaic was three times as old as the engraving. Find the present age of each.

34. The sum of the ages of Elli and Dan is 56. Four years ago Elli was 3 times as old as Dan. How old are they now?

35. A wool tapestry is 32 years older than a linen tapestry. Twenty years ago, the wool tapestry was twice as old as the linen tapestry. Find the present age of each.

36. Carolyn's age is triple her daughter's age. In eight years the sum of their ages will be 72. How old are they now?

37. Nicole is 26 years old. Emma is 2 years old. In how many years will Nicole be triple Emma's age?

38. The sum of the ages of two children is 16 years. Four years ago, the age of the older child was three times the age of the younger child. Find the present age of each child.

39. Mike is 4 years older than Ron. In two years, the sum of their ages will be 84. How old are they now?

40. A marble bust is 25 years old, and a terra-cotta bust is 85 years old. In how many years will the terra-cotta bust be three times as old as the marble bust?

1.10

Solving Linear Equations - Distance, Rate and Time

Objective: Solve distance problems by creating and solving a linear equation.

An application of linear equations can be found in distance problems. When solving distance problems we will use the relationship $rt = d$ or rate (speed) times time equals distance. For example, if a person were to travel 30 mph for 4 hours. To find the total distance we would multiply rate times time or $(30)(4) = 120$. This person travel a distance of 120 miles. The problems we will be solving here will be a few more steps than described above. So to keep the information in the problem organized we will use a table. An example of the basic structure of the table is blow:

	Rate	Time	Distance
Person 1			
Person 2			

Table 7. Structure of Distance Problem

The third column, distance, will always be filled in by multiplying the rate and time columns together. If we are given a total distance of both persons or trips we will put this information below the distance column. We will now use this table to set up and solve the following example

Example 114.

Two joggers start from opposite ends of an 8 mile course running towards each other. One jogger is running at a rate of 4 mph, and the other is running at a rate of 6 mph. After how long will the joggers meet?

	Rate	Time	Distance
Jogger 1			
Jogger 2			

The basic table for the joggers, one and two

	Rate	Time	Distance
Jogger 1	**4**		
Jogger 2	**6**		

We are given the rates for each jogger.
These are added to the table

	Rate	Time	Distance
Jogger 1	4	$\boldsymbol{t}$	
Jogger 2	6	$\boldsymbol{t}$	

We only know they both start and end at the same time. We use the variable t for both times

	Rate	Time	Distance
Jogger 1	4	t	$\boldsymbol{4t}$
Jogger 2	6	t	$\boldsymbol{6t}$
			8

The distance column is filled in by multiplying rate by time

We have **total distance**, 8 miles, under distance

$4t + 6t = 8$ The distance column gives equation by adding

$10t = 8$ Combine like terms, $4t + 6t$

$\overline{\mathbf{10}}\ \ \overline{\mathbf{10}}$ Divide both sides by 10

$t = \frac{4}{5}$ Our solution for t, $\frac{4}{5}$ hour (48 minutes)

As the example illustrates, once the table is filled in, the equation to solve is very easy to find. This same process can be seen in the following example

Example 115.

Bob and Fred start from the same point and walk in opposite directions. Bob walks 2 miles per hour faster than Fred. After 3 hours they are 30 miles apart. How fast did each walk?

	Rate	Time	Distance
Bob		**3**	
Fred		**3**	

The basic table with given times filled in
Both traveled 3 hours

	Rate	Time	Distance
Bob	$\boldsymbol{r+2}$	3	
Fred	$\boldsymbol{r}$	3	

Bob walks 2 mph faster than Fred
We know nothing about Fred, so use r for his rate
Bob is $r+2$, showing 2 mph faster

	Rate	Time	Distance
Bob	$r+2$	3	$\boldsymbol{3r+6}$
Fred	r	3	$\boldsymbol{3r}$
			30

Distance column is filled in by multiplying rate by Time. Be sure to distribute the $3(r+2)$ for Bob.

Total distance is put under distance

$3r+6+3r=30$ The distance columns is our equation, by adding

$6r+6=30$ Combine like terms $3r+3r$

$\underline{\boldsymbol{-6\ \ -6}}$ Subtract 6 from both sides

$6r=24$ The variable is multiplied by 6

$\overline{\boldsymbol{6}}\ \ \overline{\boldsymbol{6}}$ Divide both sides by 6

$r=4$ Our solution for r

	Rate
Bob	$4+2=6$
Fred	4

To answer the question completely we plug 4 in for r in the table. Bob traveled 6 miles per hour and Fred traveled 4 mph

Some problems will require us to do a bit of work before we can just fill in the cells. One example of this is if we are given a total time, rather than the individual times like we had in the previous example. If we are given total time we will write this above the time column, use t for the first person's time, and make a subtraction problem, Total $-$ t, for the second person's time. This is shown in the next example

Example 116.

Two campers left their campsite by canoe and paddled downstream at an average speed of 12 mph. They turned around and paddled back upstream at an average rate of 4 mph. The total trip took 1 hour. After how much time did the campers turn around downstream?

	Rate	Time	Distance
Down	**12**		
Up	**4**		

Basic table for down and upstream
Given rates are filled in

		1	
	Rate	Time	Distance
Down	12	$\boldsymbol{t}$	
Up	4	$\boldsymbol{1-t}$	

Total time is put above time column
As we have the total time, in the first time we have t, the second time becomes the subtraction, total $-\,t$

	Rate	Time	Distance
Down	12	t	$\mathbf{12t}$
Up	4	$1-t$	$\mathbf{4-4t}$

$=$ Distance column is found by multiplying rate by time. Be sure to distribute $4(1-t)$ for upstream. As they cover the **same distance**, $=$ is put after the down distance

$12t = 4 - 4t$	With equal sign, distance colum is equation
$\underline{+\mathbf{4t} \quad +\mathbf{4t}}$	Add $4t$ to both sides so variable is only on one side
$16t = 4$	Variable is multiplied by 16
$\overline{\mathbf{16}} \ \overline{\mathbf{16}}$	Divide both sides by 16
$t = \frac{1}{4}$	Our solution, turn around after $\frac{1}{4}$ hr (15 min)

Another type of a distance problem where we do some work is when one person catches up with another. Here a slower person has a head start and the faster person is trying to catch up with him or her and we want to know how long it will take the fast person to do this. Our startegy for this problem will be to use t for the faster person's time, and add amount of time the head start was to get the slower person's time. This is shown in the next example.

Example 117.

Mike leaves his house traveling 2 miles per hour. Joy leaves 6 hours later to catch up with him traveling 8 miles per hour. How long will it take her to catch up with him?

	Rate	Time	Distance
Mike	**2**		
Joy	**8**		

Basic table for Mike and Joy
The given rates are filled in

	Rate	Time	Distance
Mike	2	$\mathbf{t+6}$	
Joy	8	$\mathbf{t}$	

Joy, the faster person, we use t for time
Mike's time is $t+6$ showing his 6 hour head start

	Rate	Time	Distance
Mike	2	$t+6$	$\mathbf{2t+12}$
Joy	8	t	$\mathbf{8t}$

$=$ Distance column is found by multiplying the rate by time. Be sure to distribute the $2(t+6)$ for Mike As they cover the **same distance**, $=$ is put after Mike's distance

$2t + 12 = 8t$	Now the distance column is the equation
$\underline{-\mathbf{2t} \quad -\mathbf{2t}}$	Subtract $2t$ from both sides
$12 = 6t$	The variable is multiplied by 6
$\overline{\mathbf{6}} \ \overline{\mathbf{6}}$	Divide both sides by 6

$2 = t$ Our solution for t, she catches him after 2 hours

World View Note: The 10,000 race is the longest standard track event. 10,000 meters is approximately 6.2 miles. The current (at the time of printing) world record for this race is held by Ethiopian Kenenisa Bekele with a time of 26 minutes, 17.53 second. That is a rate of 12.7 miles per hour!

As these example have shown, using the table can help keep all the given information organized, help fill in the cells, and help find the equation we will solve. The final example clearly illustrates this.

Example 118.

On a 130 mile trip a car travled at an average speed of 55 mph and then reduced its speed to 40 mph for the remainder of the trip. The trip took 2.5 hours. For how long did the car travel 40 mph?

	Rate	Time	Distance
Fast	**55**		
Slow	**40**		

Basic table for fast and slow speeds
The given rates are filled in

2.5

	Rate	Time	Distance
Fast	55	$\boldsymbol{t}$	
Slow	40	$\mathbf{2.5 - \boldsymbol{t}}$	

Total time is put above the time column
As we have total time, the first time we have t
The second time is the subtraction problem $2.5 - t$

2.5

	Rate	Time	Distance
Fast	55	t	$\mathbf{55\boldsymbol{t}}$
Slow	40	$2.5 - t$	$\mathbf{100 - 40\boldsymbol{t}}$

130

Distance column is found by multiplying rate by time. Be sure to distribute $40(2.5 - t)$ for slow

Total distance is put under distance

$55t + 100 - 40t = 130$ The distance column gives our equation by adding

$15t + 100 = 130$ Combine like terms $55t - 40t$

$\underline{\mathbf{-100 \quad -100}}$ Subtract 100 from both sides

$15t = 30$ The variable is multiplied by 30

$\overline{\mathbf{15}} \quad \overline{\mathbf{15}}$ Divide both sides by 15

$t = 2$ Our solution for t.

	Time
Fast	2
Slow	$2.5 - 2 = 0.5$

To answer the question we plug 2 in for t
The car traveled 40 mph for 0.5 hours (30 minutes)

1.10 Practice - Distance, Rate, and Time Problems

1. A is 60 miles from B. An automobile at A starts for B at the rate of 20 miles an hour at the same time that an automobile at B starts for A at the rate of 25 miles an hour. How long will it be before the automobiles meet?

2. Two automobiles are 276 miles apart and start at the same time to travel toward each other. They travel at rates differing by 5 miles per hour. If they meet after 6 hours, find the rate of each.

3. Two trains travel toward each other from points which are 195 miles apart. They travel at rate of 25 and 40 miles an hour respectively. If they start at the same time, how soon will they meet?

4. A and B start toward each other at the same time from points 150 miles apart. If A went at the rate of 20 miles an hour, at what rate must B travel if they meet in 5 hours?

5. A passenger and a freight train start toward each other at the same time from two points 300 miles apart. If the rate of the passenger train exceeds the rate of the freight train by 15 miles per hour, and they meet after 4 hours, what must the rate of each be?

6. Two automobiles started at the same time from a point, but traveled in opposite directions. Their rates were 25 and 35 miles per hour respectively. After how many hours were they 180 miles apart?

7. A man having ten hours at his disposal made an excursion, riding out at the rate of 10 miles an hour and returning on foot, at the rate of 3 miles an hour.

Find the distance he rode.

8. A man walks at the rate of 4 miles per hour. How far can he walk into the country and ride back on a trolley that travels at the rate of 20 miles per hour, if he must be back home 3 hours from the time he started?

9. A boy rides away from home in an automobile at the rate of 28 miles an hour and walks back at the rate of 4 miles an hour. The round trip requires 2 hours. How far does he ride?

10. A motorboat leaves a harbor and travels at an average speed of 15 mph toward an island. The average speed on the return trip was 10 mph. How far was the island from the harbor if the total trip took 5 hours?

11. A family drove to a resort at an average speed of 30 mph and later returned over the same road at an average speed of 50 mph. Find the distance to the resort if the total driving time was 8 hours.

12. As part of his flight trainging, a student pilot was required to fly to an airport and then return. The average speed to the airport was 90 mph, and the average speed returning was 120 mph. Find the distance between the two airports if the total flying time was 7 hours.

13. A, who travels 4 miles an hour starts from a certain place 2 hours in advance of B, who travels 5 miles an hour in the same direction. How many hours must B travel to overtake A?

14. A man travels 5 miles an hour. After traveling for 6 hours another man starts at the same place, following at the rate of 8 miles an hour. When will the second man overtake the first?

15. A motorboat leaves a harbor and travels at an average speed of 8 mph toward a small island. Two hours later a cabin cruiser leaves the same harbor and travels at an average speed of 16 mph toward the same island. In how many hours after the cabin cruiser leaves will the cabin cuiser be alongside the motorboat?

16. A long distance runner started on a course running at an average speed of 6 mph. One hour later, a second runner began the same course at an average speed of 8 mph. How long after the second runner started will the second runner overtake the first runner?

17. A car traveling at 48 mph overtakes a cyclist who, riding at 12 mph, has had a 3 hour head start. How far from the starting point does the car overtake the cyclist?

18. A jet plane traveling at 600 mph overtakes a propeller-driven plane which has

had a 2 hour head start. The propeller-driven plane is traveling at 200 mph. How far from the starting point does the jet overtake the propeller-driven plane?

19. Two men are traveling in opposite directions at the rate of 20 and 30 miles an hour at the same time and from the same place. In how many hours will they be 300 miles apart?

20. Running at an average rate of 8 m/s, a sprinter ran to the end of a track and then jogged back to the starting point at an average rate of 3 m/s. The sprinter took 55 s to run to the end of the track and jog back. Find the length of the track.

21. A motorboat leaves a harbor and travels at an average speed of 18 mph to an island. The average speed on the return trip was 12 mph. How far was the island from the harbor if the total trip took 5 h?

22. A motorboat leaves a harbor and travels at an average speed of 9 mph toward a small island. Two hours later a cabin cruiser leaves the same harbor and travels at an average speed of 18 mph toward the same island. In how many hours after the cabin cruiser leaves will the cabin cruiser be alongside the motorboat?

23. A jet plane traveling at 570 mph overtakes a propeller-driven plane that has had a 2 h head start. The propeller-driven plane is traveling at 190 mph. How far from the starting point does the jet overtake the propeller-driven plane?

24. Two trains start at the same time from the same place and travel in opposite directions. If the rate of one is 6 miles per hour more than the rate of the other and they are 168 miles apart at the end of 4 hours, what is the rate of each?

25. As part of flight traning, a student pilot was required to fly to an airport and then return. The average speed on the way to the airport was 100 mph, and the average speed returning was 150 mph. Find the distance between the two airports if the total flight time was 5 h.

26. Two cyclists start from the same point and ride in opposite directions. One cyclist rides twice as fast as the other. In three hours they are 72 miles apart. Find the rate of each cyclist.

27. A car traveling at 56 mph overtakes a cyclist who, riding at 14 mph, has had a 3 h head start. How far from the starting point does the car overtake the cyclist?

28. Two small planes start from the same point and fly in opposite directions.

The first plan is flying 25 mph slower than the second plane. In two hours the planes are 430 miles apart. Find the rate of each plane.

29. A bus traveling at a rate of 60 mph overtakes a car traveling at a rate of 45 mph. If the car had a 1 h head start, how far from the starting point does the bus overtake the car?

30. Two small planes start from the same point and fly in opposite directions. The first plane is flying 25 mph slower than the second plane. In 2 h, the planes are 470 mi apart. Find the rate of each plane.

31. A truck leaves a depot at 11 A.M. and travels at a speed of 45 mph. At noon, a van leaves the same place and travels the same route at a speed of 65 mph. At what time does the van overtake the truck?

32. A family drove to a resort at an average speed of 25 mph and later returned over the same road at an average speed of 40 mph. Find the distance to the resort if the total driving time was 13 h.

33. Three campers left their campsite by canoe and paddled downstream at an average rate of 10 mph. They then turned around and paddled back upstream at an average rate of 5 mph to return to their campsite. How long did it take the campers to canoe downstream if the total trip took 1 hr?

34. A motorcycle breaks down and the rider has to walk the rest of the way to work. The motorcycle was being driven at 45 mph, and the rider walks at a speed of 6 mph. The distance from home to work is 25 miles, and the total time for the trip was 2 hours. How far did the motorcycle go before if broke down?

35. A student walks and jogs to college each day. The student averages 5 km/hr walking and 9 km/hr jogging. The distance from home to college is 8 km, and the student makes the trip in one hour. How far does the student jog?

36. On a 130 mi trip, a car traveled at an average speed of 55 mph and then reduced its speed to 40 mph for the remainder of the trip. The trip took a total of 2.5 h. For how long did the car travel at 40 mph?

37. On a 220 mi trip, a car traveled at an average speed of 50 mph and then reduced its average speed to 35 mph for the remainder of the trip. The trip took a total of 5 h. How long did the car travel at each speed?

38. An executive drove from home at an average speed of 40 mph to an airport where a helicopter was waiting. The executive boarded the helicopter and flew to the corporate offices at and average speed of 60 mph. The entire distance was 150 mi. The entire trip took 3 h. Find the distance from the airport to the corporate offices.

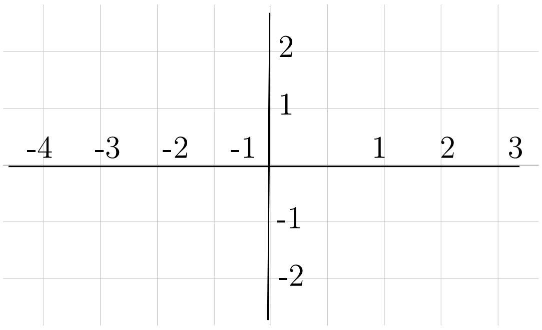
Chapter 2 : Graphing

2.1 (Part 1)

Graphing - Points and Lines

Objective: Graph points and lines using xy coordinates.

Often, to get an idea of the behavior of an equation we will make a picture that represents the solutions to the equations. A **graph** is simply a picture of the solutions to an equation. Before we spend much time on making a visual representation of an equation, we first have to understand the basis of graphing. Following is an example of what is called the coordinate plane.



The plane is divided into four sections by a horizontal number line (x-axis) and a vertical number line (y-axis). Where the two lines meet in the center is called the origin. This center origin is where $x = 0$ and $y = 0$. As we move to the right the numbers count up from zero, representing $x = 1, 2, 3 \ldots.$

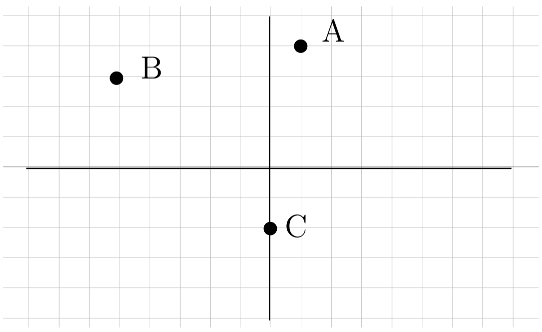
To the left the numbers count down from zero, representing $x = -1, -2, -3 \ldots.$ Similarly, as we move up the number count up from zero, $y = 1, 2, 3 \ldots.$, and as we move down count down from zero, $y = -1, -2, -3$. We can put dots on the graph which we will call points. Each point has an "address" that defines its location. The first number will be the value on the $x-$axis or horizontal number line. This is the distance the point moves left/right from the origin. The second number will represent the value on the $y-$ axis or vertical number line. This is the distance the point moves up/down from the origin. The points are given as an ordered pair (x, y).

World View Note: Locations on the globe are given in the same manner, each number is a distance from a central point, the origin which is where the prime meridian and the equator. This "origin is just off the western coast of Africa.

The following example finds the address or coordinate pair for each of several points on the coordinate plane.

Example 119.

Give the coordinates of each point.



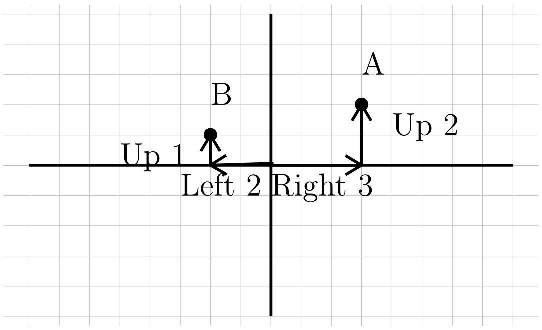
Tracing from the origin, point A is right 1, up 4. This becomes A(1, 4). Point B is left 5, up 3. Left is backwards or negative so we have B(− 5, 3). C is straight down 2 units. There is no left or right. This means we go right zero so the point is C(0, − 2).

$$A(1,4), B(-5,3), C(0,-2) \quad \text{Our Solution}$$

Just as we can give the coordinates for a set of points, we can take a set of points and plot them on the plane.

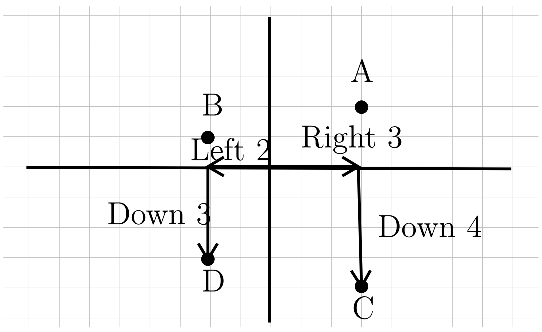
Example 120.

Graph the points $A(3,2)$, $B(-2,1)$, $C(3,-4)$, $D(-2,-3)$, $E(-3,0)$, $F(0,2)$, $G(0,0)$



The first point, A is at (3, 2) this means $x=3$ (right 3) and $y=2$ (up 2). Following these instructions, starting from the origin, we get our point.

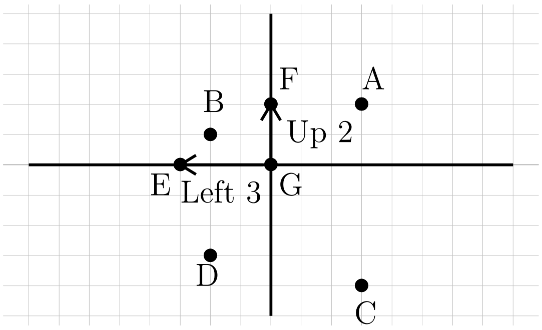
The second point, $B(-2, 1)$, is left 2 (negative moves backwards), up 1. This is also illustrated on the graph.



The third point, $C(3, -4)$ is right 3, down 4 (negative moves backwards).

The fourth point, D $(-2, -3)$ is left 2, down 3 (both negative, both move backwards)

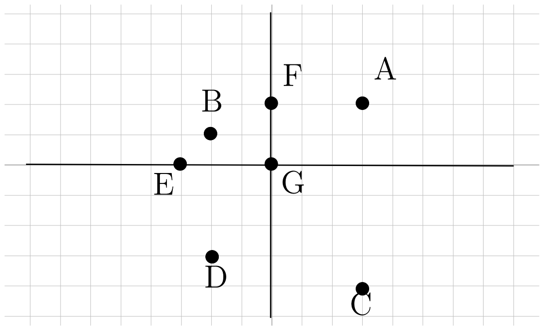
The last three points have zeros in them. We still treat these points just like the other points. If there is a zero there is just no movement.



Next is $E(-3,0)$. This is left 3 (negative is backwards), and up zero, right on the $x-$axis.

Then is $F(0, 2)$. This is right zero, and up two, right on the $y-$axis.

Finally is $G(0, 0)$. This point has no movement. Thus the point is right on the origin.



Our Solution

The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. We may have an equation such as $y = 2x - 3$. We may be interested in what type of solution are possible in this equation. We can visualize the solution by making a graph of possible x and y combinations that make this equation a true statement. We will have to start by finding possible x and y combinations. We will do this using a table of values.

Example 121.

Graph $y = 2x - 3$ We make a table of values

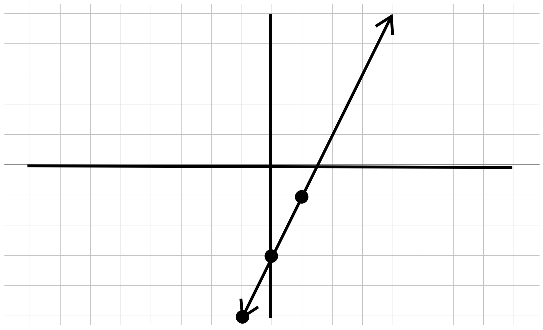
x	y
-1	
0	
1	

We will test three values for x. Any three can be used

x	y
-1	-5
0	-3
1	-1

Evaluate each by replacing x with the given value
$x = -1; y = 2(-1) - 3 = -2 - 3 = -5$
$x = 0; y = 2(0) - 3 = 0 - 3 = -3$
$x = 1; y = 2(1) - 3 = 2 - 3 = -1$

$(-1, -5), (0, -3), (1, -1)$ These then become the points to graph on our equation



Plot each point.

Once the point are on the graph, connect the dots to make a line.

The graph is our solution

What this line tells us is that any point on the line will work in the equation $y = 2x - 3$. For example, notice the graph also goes through the point $(2, 1)$. If we use $x = 2$, we should get $y = 1$. Sure enough, $y = 2(2) - 3 = 4 - 3 = 1$, just as the graph suggests. Thus we have the line is a picture of all the solutions for $y = 2x - 3$. We can use this table of values method to draw a graph of any linear equation.

Example 122.

Graph $2x - 3y = 6$	We will use a table of values

x	y
-3	
0	
3	

We will test three values for x. Any three can be used.

$2(-3) - 3y = 6$	Substitute each value in for x and solve for y
$-6 - 3y = 6$	Start with $x = -3$, multiply first
$\underline{+6 \qquad +6}$	Add 6 to both sides
$-3y = 12$	Divide both sides by -3
$\overline{-3} \quad \overline{-3}$	
$y = -4$	Solution for y when $x = -3$, add this to table

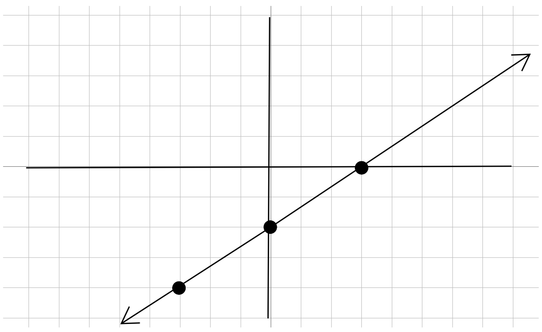
$2(0) - 3y = 6$	Next $x = 0$
$-3y = 6$	Multiplying clears the constant term
$\overline{-3} \; \overline{-3}$	Divide each side by -3
$y = -2$	Solution for y when $x = 0$, add this to table

$2(3) - 3y = 6$	Next $x = 3$
$6 - 3y = 6$	Multiply
$\underline{-6 \qquad -6}$	Subtract 9 from both sides
$-3y = 0$	Divide each side by -3
$\overline{-3} \quad \overline{-3}$	
$y = 0$	Solution for y when $x = -3$, add this to table

x	y
-3	-4
0	-2
3	0

Our completed table.

$(-3,-4),(0,2),(3,0)$ Table becomes points to graph

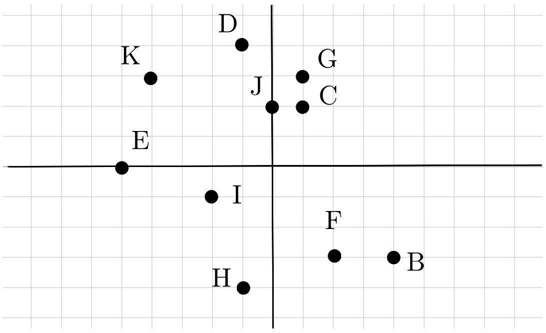


Graph points and connect dots

Our Solution

2.1 (Part 1) Practice - Points and Lines

State the coordinates of each point.

1) 

Plot each point.

2) L$(-5,5)$ K$(1,0)$ $J(-3,4)$

I$(-3,0)$ $H(-4,2)$ G(4, -2)

$F(-2,-2)$ E $(3,-2)$ D(0, 3)

C (0, 4)

Sketch the graph of each line.

3) $y=-\frac{1}{4}x-3$

4) $y=x-1$

5) $y=-\frac{5}{4}x-4$

6) $y=-\frac{3}{5}x+1$

7) $y=-4x+2$

8) $y=\frac{5}{3}x+4$

9) $y=\frac{3}{2}x-5$

10) $y=-x-2$

11) $y=-\frac{4}{5}x-3$

12) $y=\frac{1}{2}x$

13) $x+5y=-15$

14) $8x-y=5$

15) $4x+y=5$

16) $3x+4y=16$

17) $2x-y=2$

18) $7x+3y=-12$

19) $x+y=-1$

20) $3x+4y=8$

21) $x-y=-3$

22) $9x-y=-4$

2.1 (Part 2)

Horizontal and Vertical Lines

Objectives: **1. Identify and graph equations of horizontal and vertical lines**

Horizontal and Vertical Lines

We need to recognize by inspection linear equations that represent a vertical or horizontal line.

Example 7: Graph by plotting five points: y = −2.

Solution: Since the given equation does not have a variable *x*, we can rewrite it with a 0 coefficient for *x*.

$$y = 0x - 2$$

Choose any five values for *x* and see that the corresponding *y*-value is always −2.

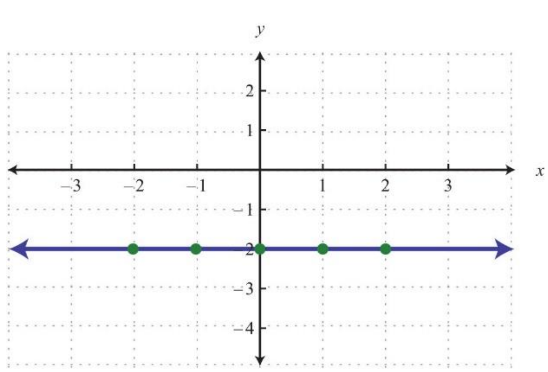
x	y
−2	−2
−1	−2
0	−2
1	−2
2	−2

$$y = 0(-2) - 2 = 0 - 2 = -2$$
$$y = 0(-1) - 2 = 0 - 2 = -2$$
$$y = 0(0) - 2 = 0 - 2 = -2$$
$$y = 0(1) - 2 = 0 - 2 = -2$$
$$y = 0(2) - 2 = 0 - 2 = -2$$

In other words, to satisfy the equation y = −2, the y-value must always be −2.

We now have five ordered pair solutions to plot {(−2, −2), (−1, −2), (0, −2), (1, −2), (2, −2)}.

Answer:



In general, the equation for a horizontal line can be written in the form y = k, where *k* represents any real number.

Example 8: Graph by plotting five points: $x = 3$.

Solution: Since the given equation does not have a variable *y*

$$x = 0y + 3$$

Choose any five values for *y* and see that the corresponding *x*-value is always 3.

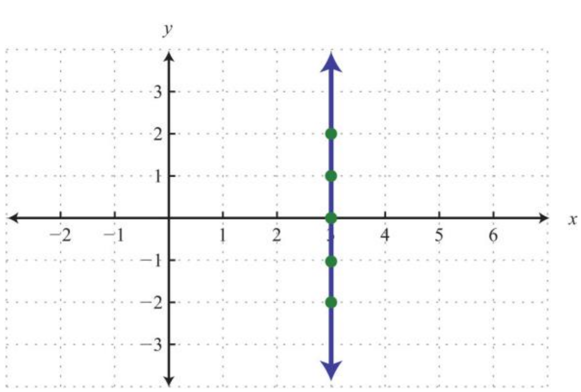
x	y
3	−2
3	−1
3	0
3	1
3	2

$$x = 0(-2) + 3 = 0 + 3 = 3$$
$$x = 0(-1) + 3 = 0 + 3 = 3$$
$$x = 0(0) + 3 = 0 + 3 = 3$$
$$x = 0(1) + 3 = 0 + 3 = 3$$
$$x = 0(2) + 3 = 0 + 3 = 3$$

In other words, to satisfy the equation $x = 3$, the *x*-value must always be 3.

We now have five ordered pair solutions to plot: {(3, −2), (3, −1), (3, 0), (3, 1), (3, 2)}.

Answer:



In general, the equation for a vertical line can be written as x=k, where *k* represents any real number.

To summarize, if *k* is a real number,

$$y = k \qquad \textit{Horizontal line}$$
$$x = k \qquad \textit{Vertical line}$$

Try this! Graph y=5 and x=−2 on the same set of axes and determine where they intersect.

Answer: (−2, 5)

Video Solution **(click to see video)**

2.1 (Part 2) Practice – Horizontal and Vertical Lines

Determine whether the given point is a solution.

1. y = 2; (−3, 2)
2. y = 4; (4, −4)
3. x = 0; (1, 0)
4. x = 3; (3, −3)

Find at least five ordered pair solutions and graph them.

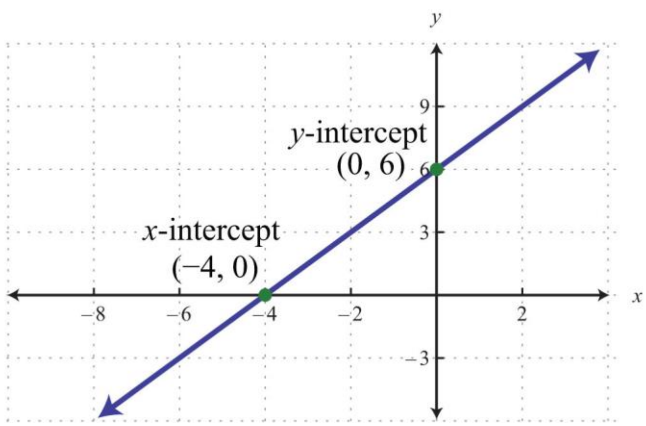
5. y=4
6. y=−10
7. x=4
8. x=−1
9. y=0
10. x=0
11. y=34
12. x=−54
13. Graph the lines y=−4 and x=2 on the same set of axes. Where do they intersect?
14. Graph the lines y=5 and x=−5 on the same set of axes. Where do they intersect?

2.1 (Part 3)

Graph Using Intercepts

Objectives: **1. Identify and find *x*- and *y*-intercepts.**
2. Graph a line using *x*- and *y*-intercepts.

Definitions: The *x*-intercept is the point where the graph of a line intersects the *x*-axis. The *y*-intercept is the point where the graph of a line intersects the *y*-axis. These points have the form (*x*, 0) and (0, *y*), respectively.



To find the *x*- and *y*-intercepts algebraically, use the fact that all *x*-intercepts have a *y*-value of zero and all *y*-intercepts have an *x*-value of zero. To find the *y*-intercept, set x=0 and determine the corresponding *y*-value. Similarly, to find the *x*-intercept, set y=0 and determine the corresponding *x*-value.

Example 1: Find the *x*- and *y*-intercepts: −3x+2y=12.

Solution: To find the *x*-intercept, set *y* = 0.

$$\begin{aligned} -3x + 2y &= 12 \\ &\downarrow \\ -3x + 2(0) &= 12 \\ -3x &= 12 \\ x &= -4 \end{aligned}$$

To find the x-intercept, set y = 0.

Therefore, the *x*-intercept is (−4, 0). To find the *y*-intercept, set *x* = 0.

$$\begin{aligned} -3x + 2y &= 12 \\ &\downarrow \\ -3(0) + 2y &= 12 \\ 2y &= 12 \\ y &= 6 \end{aligned}$$

To find the y-intercept, set x = 0.

Hence the *y*-intercept is (0, 6). Note that this linear equation is graphed above.

Answer: *x*-intercept: (−4, 0); *y*-intercept: (0, 6)

Example 2: Find the *x*- and *y*-intercepts: y=−3x+9.

Solution: Begin by finding the *x*-intercept.

$$\begin{aligned} y &= -3x+9 \quad \textit{Set } y = 0. \\ &\downarrow \\ 0 &= -3x+9 \quad \textit{Solve for } x. \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

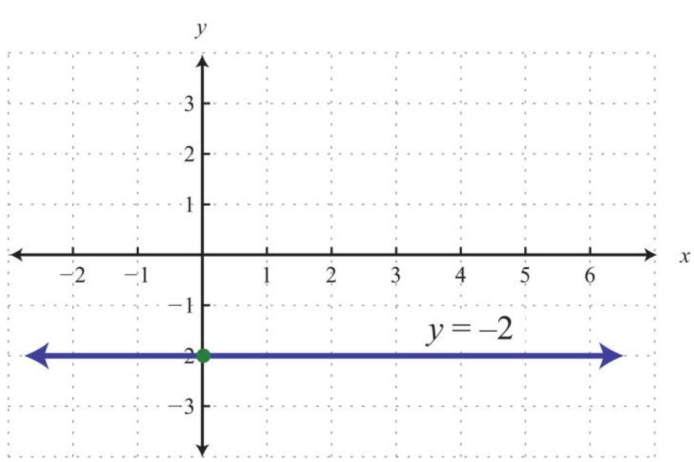
The *x*-intercept is (3, 0). Next, determine the *y*-intercept.

$$\begin{aligned} y &= -3x+9 \quad \textit{Set } x = 0. \\ &\downarrow \\ y &= -3(0)+9 \quad \textit{Solve for } y. \\ y &= 9 \end{aligned}$$

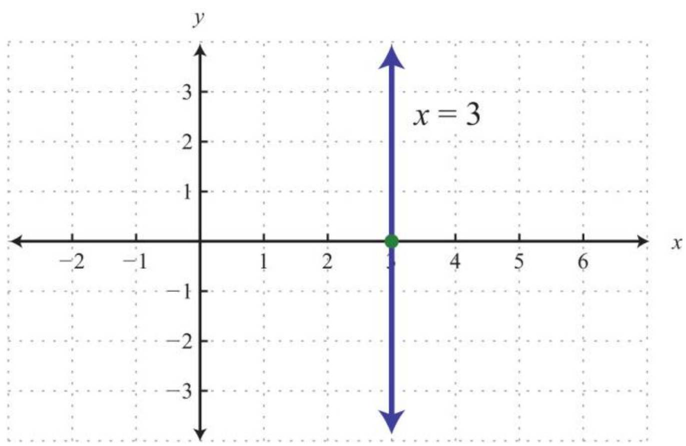
The *y*-intercept is (0, 9).

Answer: *x*-intercept: (3, 0); *y*-intercept: (0, 9)

Keep in mind that the intercepts are ordered pairs and not numbers. In other words, the *x*-intercept is not x=2 but rather (2, 0). In addition, not all graphs necessarily have both intercepts: for example,



The horizontal line graphed above has a *y*-intercept of (0, −2) and no *x*-intercept.



The vertical line graphed above has an *x*-intercept (3, 0) and no *y*-intercept.

Try this! Find the *x*- and *y*-intercepts: 4x−y=2.

Answer: *x*-intercept: (1/2, 0); *y*-intercept: (0, −2)

Video Solution (click to see video)

Graphing Lines Using Intercepts

Since two points determine a line, we can use the *x*- and *y*-intercepts to graph linear equations. We have just outlined an easy method for finding intercepts; now we outline the steps for graphing lines using the intercepts.

Example 3: Graph using intercepts: 2x−3y=12.

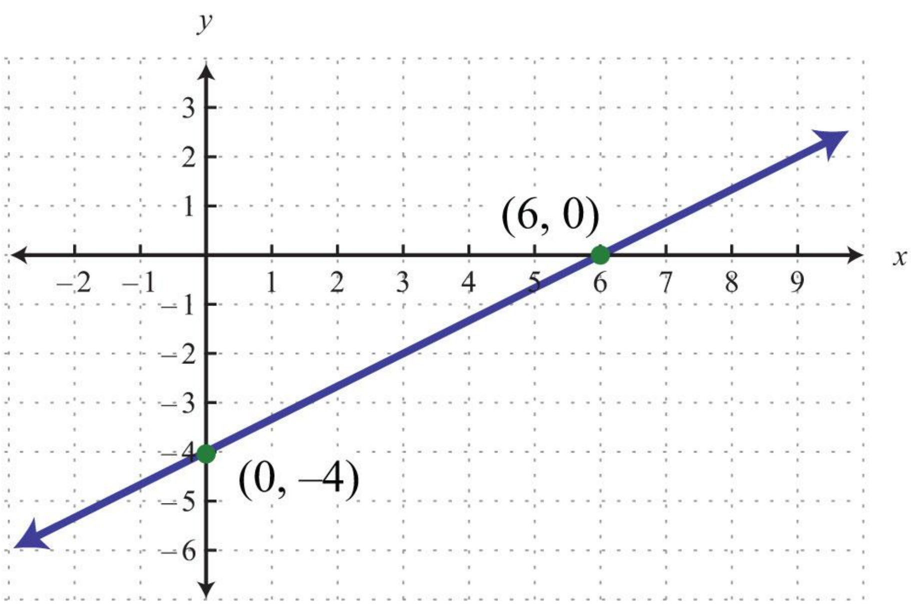
Solution:

Step 1: Find the *x*- and *y*-intercepts.

To find the *x*-intercept, set *y* = 0.	**To find the *y*-intercept, set *x* = 0.**
$2x-3y=12$	$2x-3y=12$
$2x-3(0)=12$	$2(0)-3y=12$
$2x=12$	$-3y=12$
$x=6$	$y=-4$
x-intercept: $(6,0)$	y-intercept: $(0,-4)$

Step 2: Plot the intercepts and draw the line through them. Use a straightedge to create a nice straight line. Add an arrow on either end to indicate that the line continues indefinitely in either direction.

Answer:



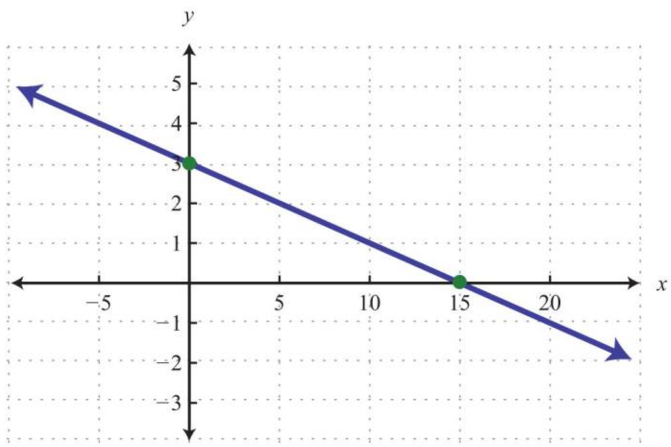
Example 4: Graph using intercepts: y=−15x+3.

Solution: Begin by determining the *x*- and *y*-intercepts.

x-intercept	*y*-intercept
$y=-\frac{1}{5}x+3$	$y=-\frac{1}{5}x+3$
$0=-\frac{1}{5}x+3$	$y=-\frac{1}{5}(0)+3$
$\frac{1}{5}x=3$	$y=3$
$5\cdot\frac{1}{5}x=5\cdot 3$	y-intercept : $(0,3)$
$x=15$	
x-intercept : $(15,0)$	

Next, graph the two points and draw a line through them with a straightedge.

Answer:



Example 5: Graph using intercepts: y=−2x.

Solution:

x-intercept	*y*-intercept
$y=-2x$	$y=-2x$
$0=-2x$	$y=-2(0)$
$\frac{0}{-2}=\frac{-2x}{-2}$	$y=0$
$0=x$	$y\text{-intercept}: (0,0)$
$x\text{-intercept}: (0,0)$	

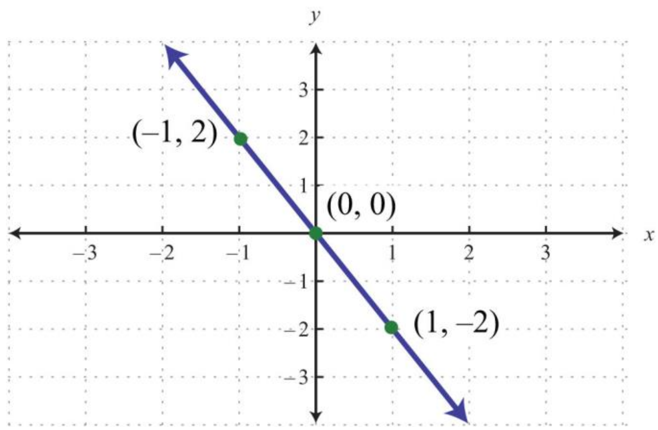
Here the *x*- and *y*-intercepts are actually the same point, the origin. We will need at least one more point so that we can graph the line. Choose any value for *x* and determine the corresponding value for *y*.

x	*y*
−1	2
1	−2

$$y=-2(-1)=2=2$$
$$y=-2(1)=-2$$

Use the ordered pair solutions (0, 0), (−1, 2), and (1, −2) to graph the line.

Answer:



To summarize, any linear equation can be graphed by finding two points and connecting them with a line drawn with a straightedge. Two important and useful points are the *x*- and *y*-intercepts; find these points by substituting *y* = 0 and *x* = 0, respectively. This method for finding intercepts will be used throughout our study of algebra.

Try this! Graph using intercepts: 3x−5y=15.

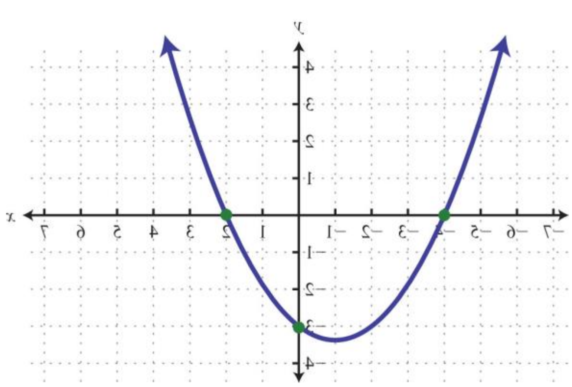
Answer: *x*-intercept: (5, 0); *y*-intercept: (0, −3)

Video Solution (click to see video)

Finding Intercepts Given the Graph

The *x*- and *y*-intercepts are important points on any graph. This chapter will focus on the graphs of linear equations. However, at this point, we can use these ideas to determine intercepts of nonlinear graphs. Remember that intercepts are ordered pairs that indicate where the graph intersects the axes.

Example 6: Find the *x*- and *y*-intercepts given the following graph:

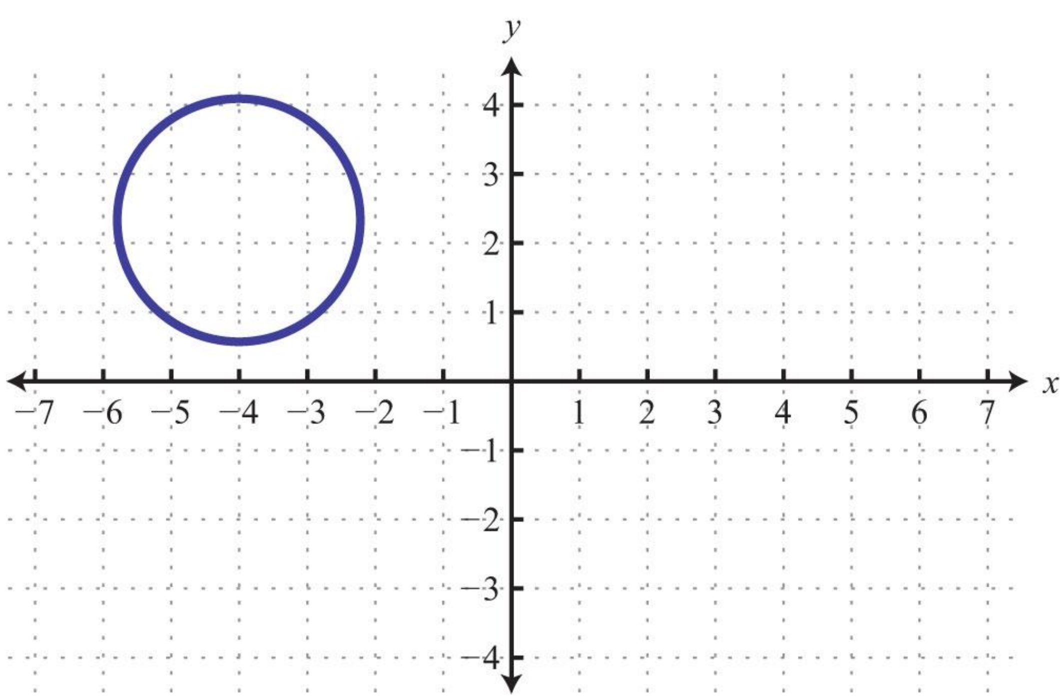


Solution: We see that the graph intersects the x-axis in two places. This graph has two x-intercepts, namely, $(-4, 0)$ and $(2, 0)$. Furthermore, the graph intersects the y-axis in one place. The only y-intercept is $(0, -3)$.

Answer: x-intercepts: $(-4, 0)$, $(2, 0)$; y-intercept: $(0, -3)$

In our study of algebra, we will see that some graphs have many intercepts. Also, we will see that some graphs do not have any.

Example 7: Given the following graph, find the x- and y-intercepts:



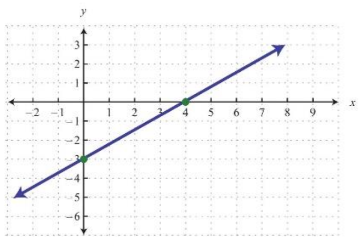
Solution: This is a graph of a circle; we can see that it does not intersect either axis. Therefore, this graph does not have any intercepts.

Answer: None

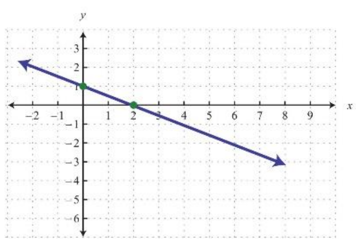
2.1 (Part 3) Practice - Graph Using Intercepts

Given the graph, find the x- and y-intercepts.

1.



2.



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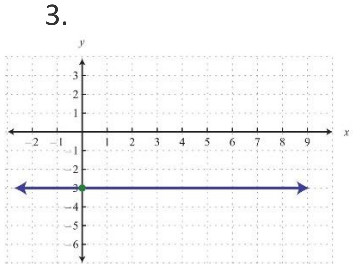

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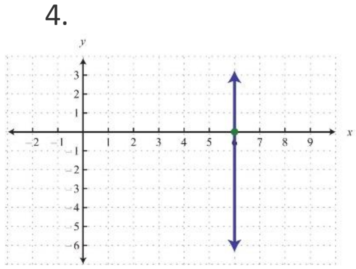

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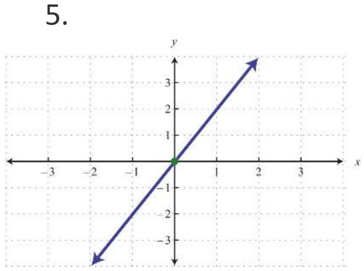
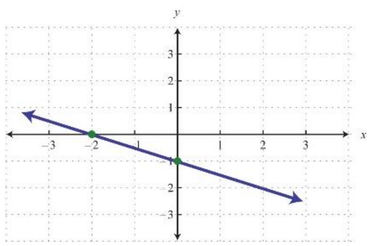


6.



Find the x- and y-intercepts.

7. $5x-4y=20$

8. $-2x+7y=-28$

9. $x-y=3$

10. $-x+y=0$

11. $3x-4y=1$

12. $-2x+5y=3$

13. $1/4x-1/3y=1$

14. $-2/5x+3/4y=2$

15. $y=6$

16. $y=-3$

17. $x=2$

18. $x=-1$

19. $y=mx+b$

20. $ax+by=c$

Find the intercepts and graph them.

21. $3x+4y=12$

22. $-2x+3y=6$

23. $5x-2y=10$

24. $-4x-8y=16$

25. $-1/2x+1/3y=1$

26. $3/4x-1/2y=-3$

27. $2x-5/2y=10$

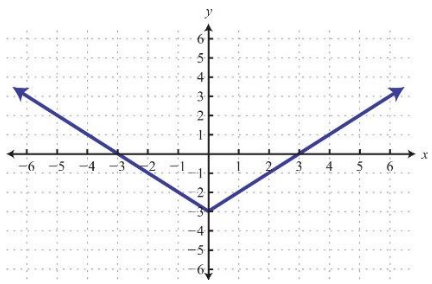
28. $2x-7/3y=-14$

29. $4x-y=-8$

30. $6x-y=6$
31. $-x+2y=1$
32. $3x+4y=6$
33. $2x+y=-1$
34. $-2x+6y=3$
35. $15x+4y=-60$
36. $-25x+3y=75$
37. $4x+2y=0$
38. $3x-y=0$
39. $-12x+6y=-4$
40. $3x+12y=-4$
41. $y=2x+4$
42. $y=-x+3$
43. $y=1/2x+1$
44. $y=2/3x-3$
45. $y=-2/5x+1$
46. $y=-5/8x-5/4$
47. $y=-7/8x-7/2$
48. $y=-x+3/2$
49. $y=3$
50. $y=32$
51. $x=5$
52. $x=-2$
53. $y=5x$
54. $y=-x$

Given the graph find the x- and y-intercepts.

55.



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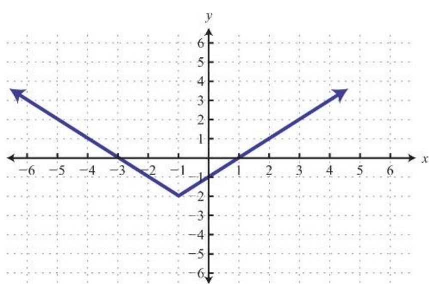

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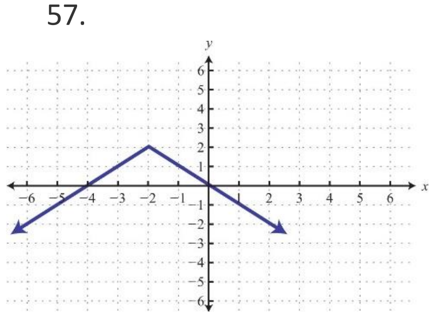

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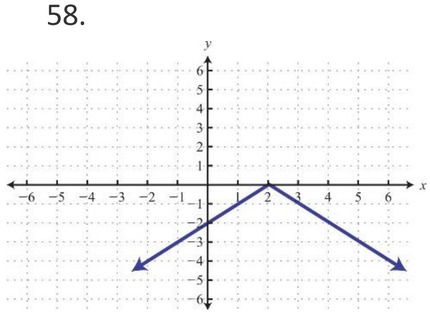

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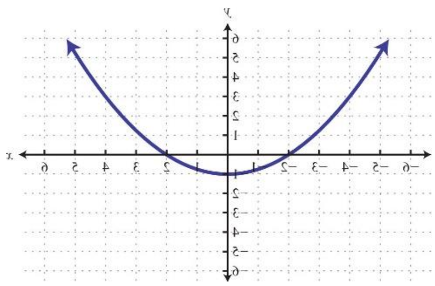

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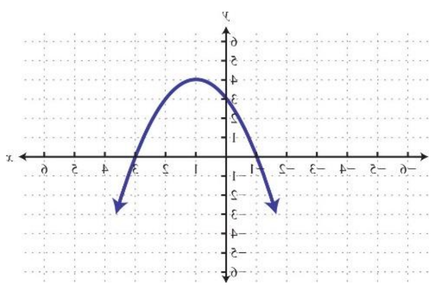

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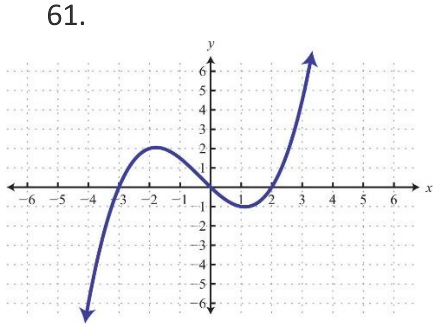
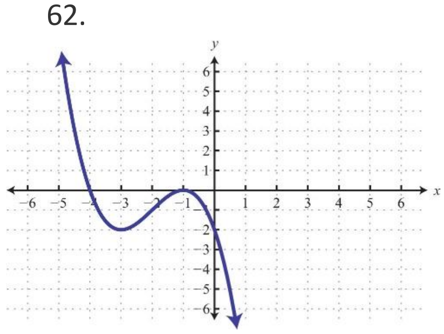
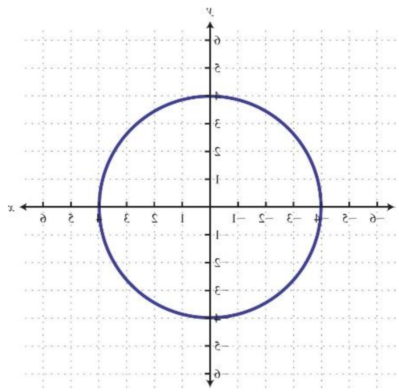


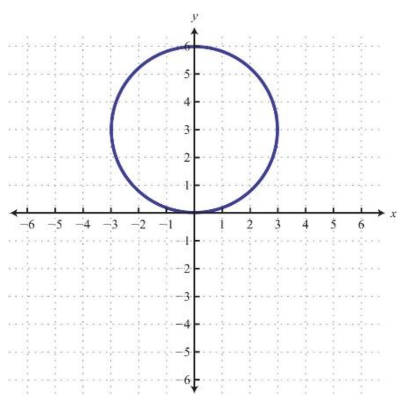
62.



63.



64.



Discussion Topics

65. What are the *x*-intercepts of the line *y* = 0?

66. What are the *y*-intercepts of the line *x* = 0?

67. Do all lines have intercepts?

68. How many intercepts can a circle have? Draw circles showing all possible numbers of intercepts.

69. Research and post the definitions of line segment, ray, and line. Why are the arrows important?

2.2

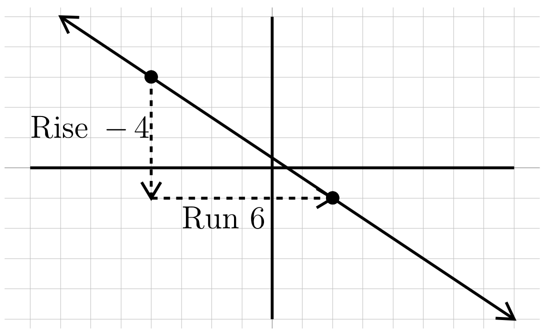
Graphing - Slope

Objective: Find the slope of a line given a graph or two points.

As we graph lines, we will want to be able to identify different properties of the lines we graph. One of the most important properties of a line is its slope. **Slope** is a measure of steepness. A line with a large slope, such as 25, is very steep. A line with a small slope, such as $\frac{1}{10}$ is very flat. We will also use slope to describe the direction of the line. A line that goes up from left to right will have a positive slope and a line that goes down from left to right will have a negative slope.

As we measure steepness we are interested in how fast the line rises compared to how far the line runs. For this reason we will describe slope as the fraction $\frac{\text{rise}}{\text{run}}$. Rise would be a vertical change, or a change in the y-values. Run would be a horizontal change, or a change in the x-values. So another way to describe slope would be the fraction $\frac{\text{change in } y}{\text{change in } x}$. It turns out that if we have a graph we can draw vertical and horiztonal lines from one point to another to make what is called a slope triangle. The sides of the slope triangle give us our slope. The following examples show graphs that we find the slope of using this idea.

Example 123.

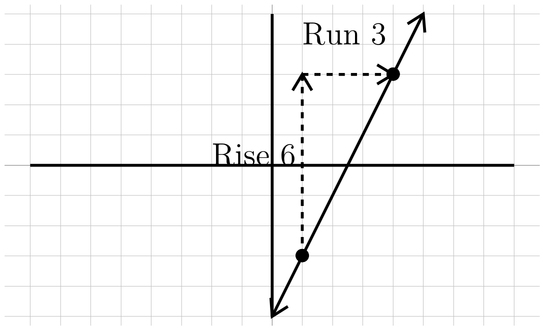


To find the slope of this line we will consider the rise, or verticle change and the run or horizontal change. Drawing these lines in makes a slope triangle that we can use to count from one point to the next the graph goes down 4, right 6. This is rise -4, run 6. As a fraction it would be, $\frac{-4}{6}$. Reduce the fraction to get $-\frac{2}{3}$.

$$-\frac{2}{3} \quad \text{Our Solution}$$

World View Note: When French mathematicians Rene Descartes and Pierre de Fermat first developed the coordinate plane and the idea of graphing lines (and other functions) the y-axis was not a verticle line!

Example 124.

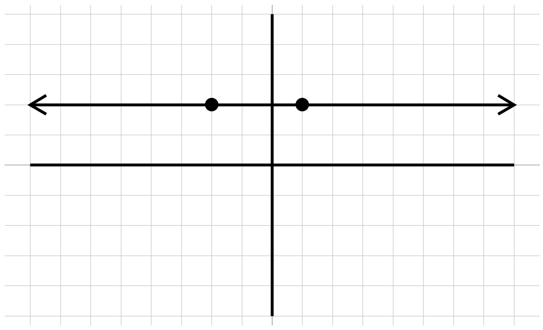


To find the slope of this line, the rise is up 6, the run is right 3. Our slope is then written as a fraction, $\frac{\text{rise}}{\text{run}}$ or $\frac{6}{3}$. This fraction reduces to 2. This will be our slope.

2 Our Solution

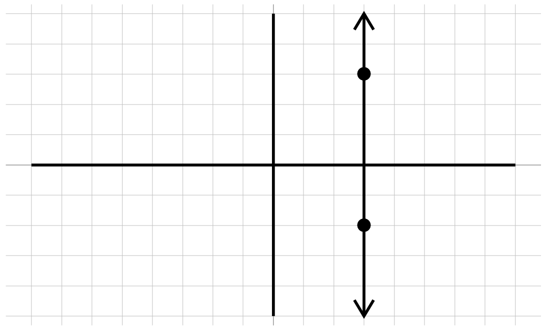
There are two special lines that have unique slopes that we need to be aware of. They are illustrated in the following example.

Example 125.



In this graph there is no rise, but the run is 3 units. This slope becomes

$\frac{0}{3} = 0$. This line, and all horizontal lines have *a* zero slope.



This line has a rise of 5, but no run. The slope becomes $\frac{5}{0} =$ undefined. This line, and all vertical lines, have no slope.

As you can see there is a big difference between having a zero slope and having no slope or undefined slope. Remember, slope is a measure of steepness. The first slope is not steep at all, in fact it is flat. Therefore it has a zero slope. The second slope can't get any steeper. It is so steep that there is no number large enough to express how steep it is. This is an undefined slope.

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y values, we can calculate this by subtracting the y values of a point. Similarly, if run is a change in the x values, we can calculate this by subtracting the x values of a point. In this way we get the following equation for slope.

$$\textbf{The slope of } \boldsymbol{a} \textbf{ line through } \boldsymbol{(x_1, y_1)} \textbf{ and } \boldsymbol{(x_2, y_2)} \textbf{ is } \boldsymbol{\frac{y_2 - y_1}{x_2 - x_1}}$$

When mathematicians began working with slope, it was called the modular slope. For this reason we often represent the slope with the variable m. Now we have the following for slope.

$$\textbf{Slope} = \boldsymbol{m} = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } \boldsymbol{y}}{\textbf{change in } \boldsymbol{x}} = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}}$$

As we subtract the y values and the x values when calculating slope it is important we subtract them in the same order. This process is shown in the following examples.

Example 126.

$$\begin{array}{rl}
\text{Find the slope between } (-4,3) \text{ and } (2,-9) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-9-3}{2-(-4)} & \text{Simplify} \\
m = \dfrac{-12}{6} & \text{Reduce} \\
m = -2 & \text{Our Solution}
\end{array}$$

Example 127.

$$\begin{array}{rl}
\text{Find the slope between } (4, \quad 6) \text{ and } (2,-1) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-1-6}{2-4} & \text{Simplify} \\
m = \dfrac{-7}{-2} & \text{Reduce, dividing by } -1 \\
m = \dfrac{7}{2} & \text{Our Solution}
\end{array}$$

We may come up against a problem that has a zero slope (horiztonal line) or no slope (vertical line) just as with using the graphs.

Example 128.

$$\begin{array}{rl}
\text{Find the slope between } (-4,-1) \text{ and } (-4,-5) & \text{Identify } x_1, y_1, x_2, y_2
\end{array}$$

$$\begin{array}{rl}(x_1\ ,y_1)\text{ and }\ (x_2,\ \ y_2) & \text{Use slope formula, } m=\dfrac{y_2-y_1}{x_2-x_1}\\ m=\dfrac{-5-(-1)}{-4-(-4)} & \text{Simplify}\\ m=\dfrac{-4}{0} & \text{Can}'t\text{ divide by zero, undefined}\\ m=\text{no slope} & \text{Our Solution}\end{array}$$

Example 129.

$$\begin{array}{rl}\text{Find the slope between }(3,1)\text{ and }(-2,1) & \text{Identify } x_1,y_1,x_2,y_2\\ (x_1,y_1)\text{ and }(x_2,y_2) & \text{Use slope formula, } m=\dfrac{y_2-y_1}{x_2-x_1}\\ m=\dfrac{1-1}{-2-3} & \text{Simplify}\\ m=\dfrac{0}{-5} & \text{Reduce}\\ m=0 & \text{Our Solution}\end{array}$$

Again, there is a big difference between no slope and a zero slope. Zero is an integer and it has a value, the slope of a flat horizontal line. No slope has no value, it is undefined, the slope of a vertical line.

Using the slope formula we can also find missing points if we know what the slope is. This is shown in the following two examples.

Example 130.

Find the value of y between the points $(2,y)$ and $(5,-1)$ with slope -3

$$\begin{array}{rl}m=\dfrac{y_2-y_1}{x_2-x_1} & \text{We will plug values into slope formula}\\ -3=\dfrac{-1-y}{5-2} & \text{Simplify}\\ -3=\dfrac{-1-y}{3} & \text{Multiply both sides by 3}\\ -3(3)=\dfrac{-1-y}{3}(3) & \text{Simplify}\\ -9=-1-y & \text{Add 1 to both sides}\\ \underline{+1\quad +1} & \\ -8=-y & \text{Divide both sides by }-1\\ \overline{-1}\quad\overline{-1} & \\ 8=y & \text{Our Solution}\end{array}$$

Example 131.

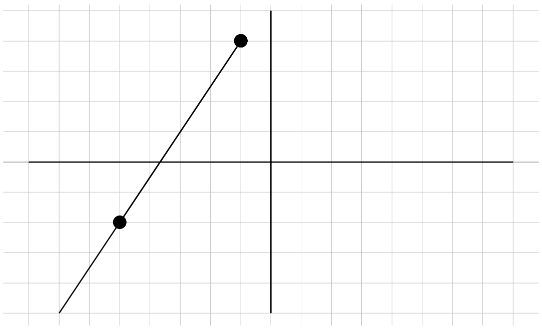
Find the value of x between the points $(-3,2)$ and $(x,6)$ with slope $\frac{2}{5}$

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{We will plug values into slope formula}\\
\frac{2}{5} &= \frac{6-2}{x-(-3)} && \text{Simplify}\\
\frac{2}{5} &= \frac{4}{x+3} && \text{Multiply both sides by } (x+3)\\
\frac{2}{5}(x+3) &= 4 && \text{Multiply by 5 to clear fraction}\\
(5)\frac{2}{5}(x+3) &= 4(5) && \text{Simplify}\\
2(x+3) &= 20 && \text{Distribute}\\
2x+6 &= 20 && \text{Solve.}\\
\underline{-6 \quad -6} & && \text{Subtract 6 from both sides}\\
2x &= 14 && \text{Divide each side by 2}\\
\overline{2} \quad & \overline{2} && \\
x &= 7 && \text{Our Solution}
\end{aligned}
$$

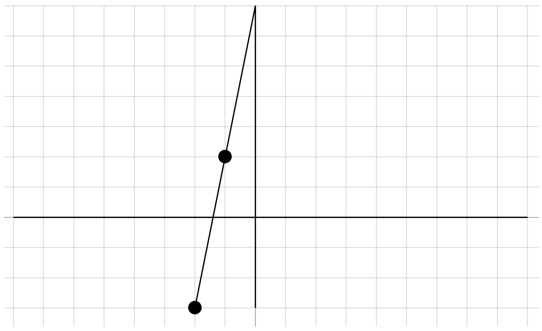
2.2 Practice - Slope

Find the slope of each line.

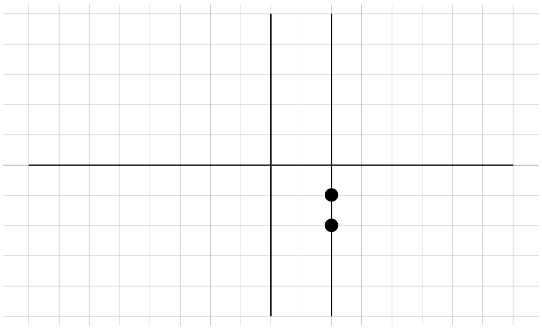
1)



2)



3)



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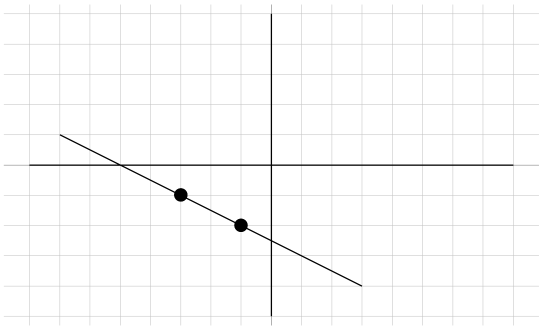

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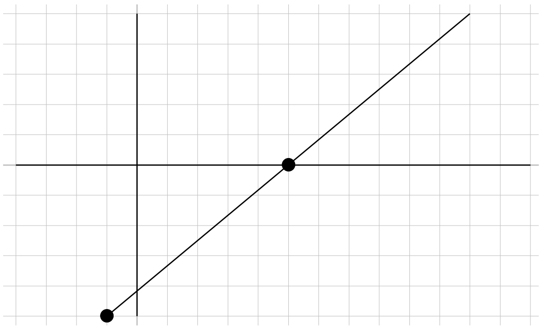

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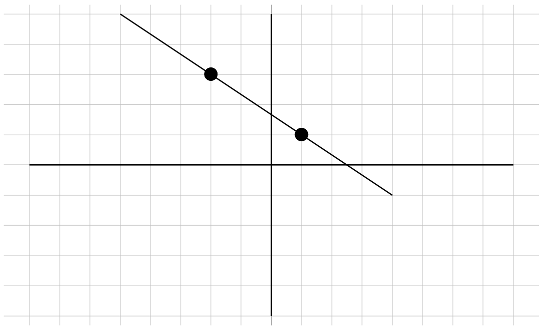

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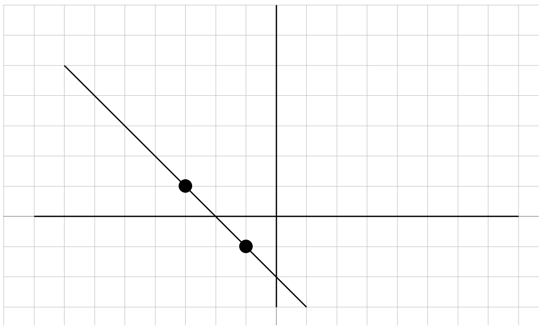
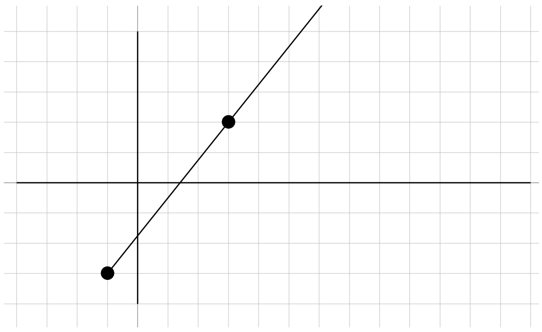
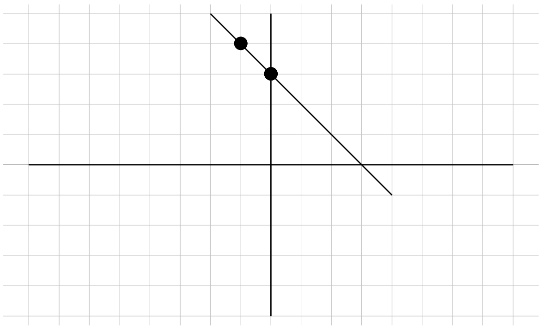


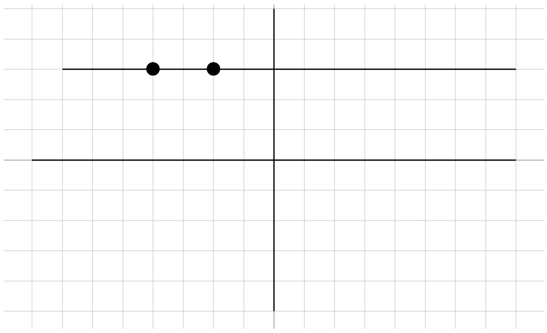
8)



9)



10)



Find the slope of the line through each pair of points.

11) $(-2,10),(-2,-15)$

12) $(1,2),(-6,-14)$

13) $(-15,10),(16,-7)$

14) $(13,-2),(7,7)$

15) $(10,18),(-11,-10)$

16) $(-3,6),(-20,13)$

17) $(-16,-14),(11,-14)$

18) $(13,15),(2,10)$

19) $(-4,14),(-16,8)$

20) $(9,-6),(-7,-7)$

21) $(12,-19),(6,14)$

22) $(-16,2),(15,-10)$

23) $(-5,-10),(-5,20)$

24) $(8,11),(-3,-13)$

25) $(-17,19),(10,-7)$

26) $(11,-2),(1,17)$

27) $(7,-14),(-8,-9)$

28) $(-18,-5),(14,-3)$

29) $(-5,7),(-18,14)$

30) $(19,15),(5,11)$

Find the value of x or y so that the line through the points has the given slope.

31) $(2,6)$ and $(x,2)$; slope: $\frac{4}{7}$

32) $(8,y)$ and $(-2,4)$; slope: $-\frac{1}{5}$

33) $(-3,-2)$ and $(x,6)$; slope: $-\frac{8}{5}$

34) $(-2,y)$ and $(2,4)$; slope: $\frac{1}{4}$

35) $(-8,y)$ and $(-1,1)$; slope: $\frac{6}{7}$

36) $(x,-1)$ and $(-4,6)$; slope: $-\frac{7}{10}$

37) $(x,-7)$ and $(-9,-9)$; slope: $\frac{2}{5}$

38) $(2,-5)$ and $(3,y)$; slope: 6

39) $(x,5)$ and $(8,0)$; slope: $-\frac{5}{6}$

40) $(6,2)$ and $(x,6)$; slope: $-\frac{4}{5}$

2.3

Graphing - Slope-Intercept Form

Objective: Give the equation of a line with a known slope and y-intercept.

When graphing a line we found one method we could use is to make a table of values. However, if we can identify some properties of the line, we may be able to make a graph much quicker and easier. One such method is finding the slope and the y-intercept of the equation. The slope can be represented by m and the y-intercept, where it crosses the axis and $x = 0$, can be represented by $(0, b)$ where b is the value where the graph crosses the vertical y-axis. Any other point on the line can be represented by (x, y). Using this information we will look at the slope formula and solve the formula for y.

Example 132.

$$\begin{array}{rl} m, (0, b), (x, y) & \text{Using the slope formula gives:} \\ \dfrac{y - b}{x - 0} = m & \text{Simplify} \\ \dfrac{y - b}{x} = m & \text{Multiply both sides by } x \\ y - b = mx & \text{Add } b \text{ to both sides} \\ \underline{+b \quad +b} & \\ y = mx + b & \text{Our Solution} \end{array}$$

This equation, $y = mx + b$ can be thought of as the equation of any line that as a slope of m and a y-intercept of b. This formula is known as the slope-intercept equation.

$$\textbf{Slope} - \textbf{Intercept Equation: } \boldsymbol{y = mx + b}$$

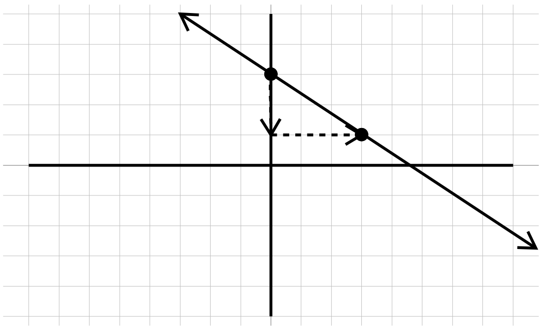
If we know the slope and the y-intercept we can easily find the equation that represents the line.

Example 133.

$$\begin{array}{rl} \text{Slope} = \dfrac{3}{4}, y - \text{intercept} = -3 & \text{Use the slope} - \text{intercept equation} \\ y = mx + b & m \text{ is the slope, } b \text{ is the } y - \text{intercept} \\ y = \dfrac{3}{4}x - 3 & \text{Our Solution} \end{array}$$

We can also find the equation by looking at a graph and finding the slope and y-intercept.

Example 134.



Identify the point where the graph crosses the y-axis (0,3). This means the y-intercept is 3.

Idenfity one other point and draw a slope triangle to find the slope. The slope is $-\frac{2}{3}$

$y = mx + b$ Slope-intercept equation

$$y = -\frac{2}{3}x + 3 \quad \text{Our Solution}$$

We can also move the opposite direction, using the equation identify the slope and y-intercept and graph the equation from this information. However, it will be important for the equation to first be in slope intercept form. If it is not, we will have to solve it for y so we can identify the slope and the y-intercept.

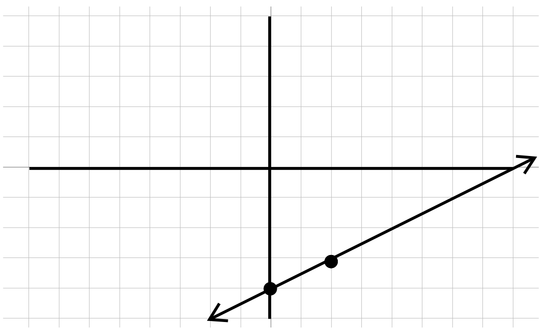
Example 135.

$$\begin{array}{rl} \text{Write in slope} - \text{intercept form:} \; 2x - 4y = 6 & \text{Solve for } y \\ -2x \qquad -2x & \text{Subtract } 2x \text{ from both sides} \\ -4y = -2x + 6 & \text{Put } x \text{ term first} \\ \overline{-4} \qquad \overline{-4} \; \overline{-4} & \text{Divide each term by } -4 \\ y = \frac{1}{2}x - \frac{3}{2} & \text{Our Solution} \end{array}$$

Once we have an equation in slope-intercept form we can graph it by first plotting the y-intercept, then using the slope, find a second point and connecting the dots.

Example 136.

$$\begin{array}{rl} \text{Graph } y = \frac{1}{2}x - 4 & \text{Recall the slope} - \text{intercept formula} \\ y = mx + b & \text{Idenfity the slope, } m\text{, and the } y - \text{intercept, } b \\ m = \frac{1}{2}, b = -4 & \text{Make the graph} \end{array}$$



Starting with a point at the y-intercept of -4,

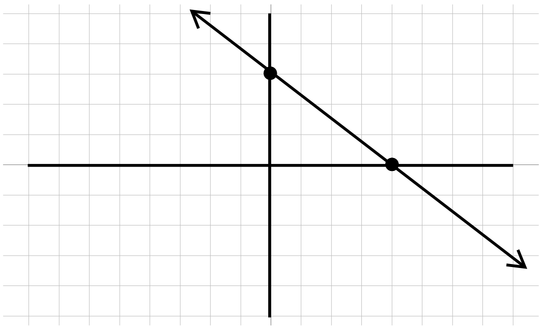
Then use the slope $\frac{\text{rise}}{\text{run}}$, so we will rise 1 unit and run 2 units to find the next point.

Once we have both points, connect the dots to get our graph.

World View Note: Before our current system of graphing, French Mathematician Nicole Oresme, in 1323 sugggested graphing lines that would look more like a bar graph with a constant slope!

Example 137.

$$\begin{array}{rl}
\text{Graph } 3x+4y=12 & \text{Not in slope intercept form} \\
-3x \qquad -3x & \text{Subtract } 3x \text{ from both sides} \\
4y=-3x+12 & \text{Put the } x \text{ term first} \\
\overline{4} \qquad \overline{4} \quad \overline{4} & \text{Divide each term by } 4 \\
y=-\frac{3}{4}x+3 & \text{Recall slope} - \text{intercept equation} \\
y=mx+b & \text{Idenfity } m \text{ and } b \\
m=-\frac{3}{4}, b=3 & \text{Make the graph}
\end{array}$$



Starting with a point at the y-intercept of 3,

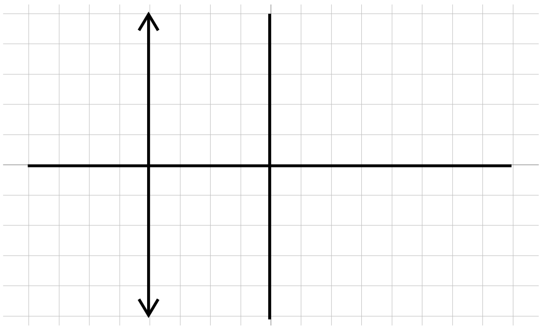
Then use the slope $\frac{\text{rise}}{\text{run}}$, but its negative so it will go downhill, so we will drop 3 units and run 4 units to find the next point.

Once we have both points, connect the dots to get our graph.

We want to be very careful not to confuse using slope to find the next point with use a coordinate such as $(4, -2)$ to find an individule point. Coordinates such as $(4, -2)$ start from the origin and move horizontally first, and vertically second. Slope starts from a point on the line that could be anywhere on the graph. The numerator is the vertical change and the denominator is the horizontal change.

Lines with zero slope or no slope can make a problem seem very different. Zero slope, or horiztonal line, will simply have a slope of zero which when multiplied by x gives zero. So the equation simply becomes $y=b$ or y is equal to the y-coordinate of the graph. If we have no slope, or a vertical line, the equation can't be written in slope intercept at all because the slope is undefined. There is no y in these equations. We will simply make x equal to the x-coordinate of the graph.

Example 138.



Give the equation of the line in the graph.

Because we have a vertical line and no slope there is no slope-intercept equation we can use. Rather we make x equal to the x-coordinate of -4

$x=-4$ Our Solution

2.3 Practice - Slope-Intercept

Write the slope-intercept form of the equation of each line given the slope and the y-intercept.

1) Slope $= 2$, y-intercept $= 5$

2) Slope $= -6$, y-intercept $= 4$

3) Slope $= 1$, y-intercept $= -4$

4) Slope $= -1$, y-intercept $= -2$

5) Slope $= -\frac{3}{4}$, y-intercept $= -1$

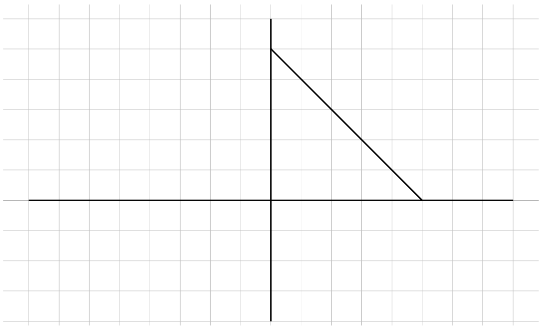
6) Slope $= -\frac{1}{4}$, y-intercept $= 3$

7) Slope $= \frac{1}{3}$, y-intercept $= 1$

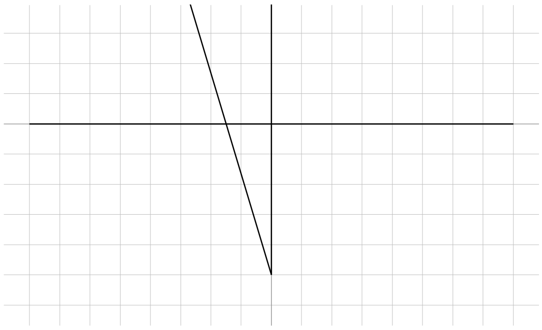
8) Slope $= \frac{2}{5}$, y-intercept $= 5$

Write the slope-intercept form of the equation of each line.

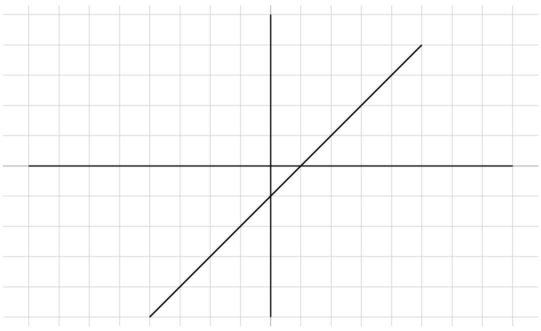
9)



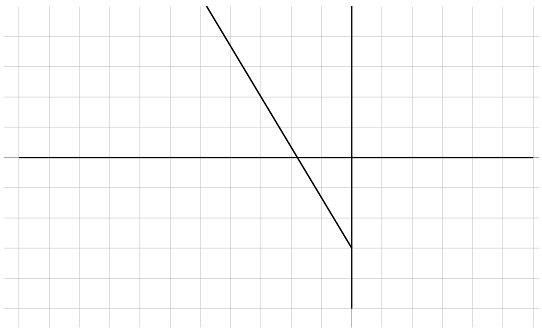
10)



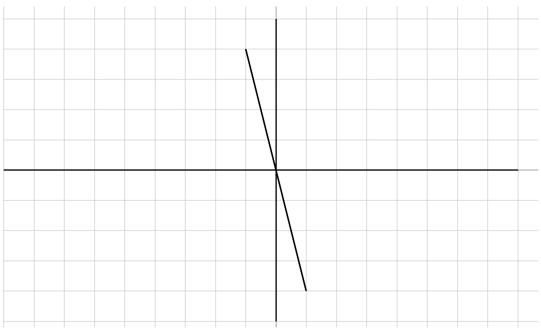
11)



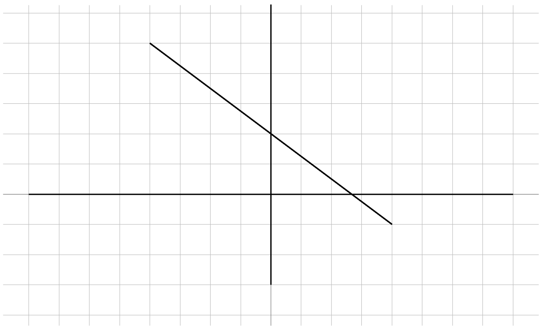
12)



13)



14)



15) $x+10y=-37$

16) $x-10y=3$

17) $2x+y=-1$

18) $6x-11y=-70$

19) $7x-3y=24$

20) $4x+7y=28$

21) $x=-8$

22) $x-7y=-42$

23) $y-4=-(x+5)$

24) $y-5=\frac{5}{2}(x-2)$

25) $y-4=4(x-1)$

26) $y-3=-\frac{2}{3}(x+3)$

27) $y+5=-4(x-2)$

28) $0=x-4$

29) $y+1=-\frac{1}{2}(x-4)$

30) $y+2=\frac{6}{5}(x+5)$

Sketch the graph of each line.

31) $y=\frac{1}{3}x+4$

32) $y=-\frac{1}{5}x-4$

33) $y=\frac{6}{5}x-5$

34) $y=-\frac{3}{2}x-1$

35) $y=\frac{3}{2}x$

36) $y=-\frac{3}{4}x+1$

37) $x-y+3=0$

38) $4x+5=5y$

39) $-y-4+3x=0$

40) $-8=6x-2y$

41) $-3y=-5x+9$

42) $-3y=3-\frac{3}{2}x$

2.4

Graphing - Point-Slope Form

Objective: Give the equation of a line with a known slope and point.

The slope-intercept form has the advantage of being simple to remember and use, however, it has one major disadvantage: we must know the y-intercept in order to use it! Generally we do not know the y-intercept, we only know one or more points (that are not the y-intercept). In these cases we can't use the slope intercept equation, so we will use a different more flexible formula. If we let the slope of an equation be m, and a specific point on the line be (x_1, y_1), and any other point on the line be (x, y). We can use the slope formula to make a second equation.

Example 139.

$$\begin{array}{rl} m, (x_1, y_1), (x, y) & \text{Recall slope formula} \\ \dfrac{y_2 - y_1}{x_2 - x_1} = m & \text{Plug in values} \\ \dfrac{y - y_1}{x - x_1} = m & \text{Multiply both sides by } (x - x_1) \\ y - y_1 = m(x - x_1) & \text{Our Solution} \end{array}$$

If we know the slope, m of an equation and any point on the line (x_1, y_1) we can easily plug these values into the equation above which will be called the point-slope formula.

$$\textbf{Point} - \textbf{Slope Formula: } \boldsymbol{y - y_1 = m(x - x_1)}$$

Example 140.

Write the equation of the line through the point $(3, -4)$ with a slope of $\frac{3}{5}$.

$$\begin{array}{rl} y - y_1 = m(x - x_1) & \text{Plug values into point} - \text{slope formula} \\ y - (-4) = \dfrac{3}{5}(x - 3) & \text{Simplify signs} \\ y + 4 = \dfrac{3}{5}(x - 3) & \text{Our Solution} \end{array}$$

Often, we will prefer final answers be written in slope intercept form. If the directions ask for the answer in slope-intercept form we will simply distribute the slope, then solve for y.

Example 141.

Write the equation of the line through the point $(-6,2)$ with a slope of $-\frac{2}{3}$ in slope-intercept form.

$$\begin{array}{rl} y-y_1=m(x-x_1) & \text{Plug values into point}-\text{slope formula} \\ y-2=-\dfrac{2}{3}(x-(-6)) & \text{Simplify signs} \\ y-2=-\dfrac{2}{3}(x+6) & \text{Distribute slope} \\ y-2=-\dfrac{2}{3}x-4 & \text{Solve for } y \\ \underline{+2 \qquad\quad +2} & \\ y=-\dfrac{2}{3}x-2 & \text{Our Solution} \end{array}$$

An important thing to observe about the point slope formula is that the operation between the x's and y's is subtraction. This means when you simplify the signs you will have the opposite of the numbers in the point. We need to be very careful with signs as we use the point-slope formula.

In order to find the equation of a line we will always need to know the slope. If we don't know the slope to begin with we will have to do some work to find it first before we can get an equation.

Example 142.

Find the equation of the line through the points $(-2,5)$ and $(4,-3)$.

$$\begin{array}{rl} m=\dfrac{y_2-y_1}{x_2-x_1} & \text{First we must find the slope} \\ m=\dfrac{-3-5}{4-(-2)}=\dfrac{-8}{6}=-\dfrac{4}{3} & \text{Plug values in slope formula and evaluate} \\ y-y_1=m(x-x_1) & \text{With slope and either point, use point}-\text{slope formula} \\ y-5=-\dfrac{4}{3}(x-(-2)) & \text{Simplify signs} \\ y-5=-\dfrac{4}{3}(x+2) & \text{Our Solution} \end{array}$$

Example 143.

Find the equation of the line through the points $(-3,4)$ and $(-1,-2)$ in slope-intercept form.

$$\begin{array}{rl}
m=\dfrac{y_2-y_1}{x_2-x_1} & \text{First we must find the slope}\\
m=\dfrac{-2-4}{-1-(-3)}=\dfrac{-6}{2}=-3 & \text{Plug values in slope formula and evaluate}\\
y-y_1=m(x-x_1) & \text{With slope and either point, point} - \text{slope formula}\\
y-4=-3(x-(-3)) & \text{Simplify signs}\\
y-4=-3(x+3) & \text{Distribute slope}\\
y-4=-3x-9 & \text{Solve for } y\\
\underline{+4 \qquad\quad +4} & \text{Add 4 to both sides}\\
y=-3x-5 & \text{Our Solution}
\end{array}$$

Example 144.

Find the equation of the line through the points $(6,-2)$ and $(-4,1)$ in slope-intercept form.

$$\begin{array}{rl}
m=\dfrac{y_2-y_1}{x_2-x_1} & \text{First we must find the slope}\\
m=\dfrac{1-(-2)}{-4-6}=\dfrac{3}{-10}=-\dfrac{3}{10} & \text{Plug values into slope formula and evaluate}\\
y-y_1=m(x-x_1) & \text{Use slope and either point, use point} - \text{slope formula}\\
y-(-2)=-\dfrac{3}{10}(x-6) & \text{Simplify signs}\\
y+2=-\dfrac{3}{10}(x-6) & \text{Distribute slope}\\
y+2=-\dfrac{3}{10}x+\dfrac{9}{5} & \text{Solve for } y\text{. Subtract 2 from both sides}\\
\underline{-2 \qquad\qquad -\dfrac{10}{5}} & \text{Using } \dfrac{10}{5} \text{ on right so we have } a \text{ common denominator}\\
y=-\dfrac{3}{10}x-\dfrac{1}{5} & \text{Our Solution}
\end{array}$$

World View Note: The city of Konigsberg (now Kaliningrad, Russia) had a river that flowed through the city breaking it into several parts. There were 7 bridges that connected the parts of the city. In 1735 Leonhard Euler considered the question of whether it was possible to cross each bridge exactly once and only once. It turned out that this problem was impossible, but the work laid the foundation of what would become graph theory.

2.4 Practice - Point-Slope Form

Write the point-slope form of the equation of the line through the given point with the given slope.

1) through $(2,3)$, slope $=$ undefined

2) through $(1,2)$, slope $=$ undefined

3) through $(2,2)$, slope $=\frac{1}{2}$

4) through $(2,1)$, slope $=-\frac{1}{2}$

5) through $(-1,-5)$, slope $=9$

6) through $(2,-2)$, slope $=-2$

7) through $(-4,1)$, slope $=\frac{3}{4}$

8) through $(4,-3)$, slope $=-2$

9) through $(0,-2)$, slope $=-3$

10) through $(-1,1)$, slope $=4$

11) through $(0,-5)$, slope $=-\frac{1}{4}$

12) through $(0,2)$, slope $=-\frac{5}{4}$

13) through $(-5,-3)$, slope $=\frac{1}{5}$

14) through $(-1,-4)$, slope $=-\frac{2}{3}$

15) through $(-1,4)$, slope $=-\frac{5}{4}$

16) through $(1,-4)$, slope $=-\frac{3}{2}$

Write the slope-intercept form of the equation of the line through the given point with the given slope.

17) through: $(-1,-5)$, slope $=2$

18) through: $(2,-2)$, slope $=-2$

19) through: $(5,-1)$, slope $=-\frac{3}{5}$

20) through: $(-2,-2)$, slope $=-\frac{2}{3}$

21) through: $(-4,1)$, slope $=\frac{1}{2}$

22) through: $(4,-3)$, slope $=-\frac{7}{4}$

23) through: $(4,-2)$, slope $=-\frac{3}{2}$

24) through: $(-2,0)$, slope $=-\frac{5}{2}$

25) through: $(-5,-3)$, slope $=-\frac{2}{5}$

26) through: $(3,3)$, slope $=\frac{7}{3}$

27) through: $(2,-2)$, slope $=1$

28) through: $(-4,-3)$, slope $=0$

29) through: $(-3,4)$, slope $=$ undefined

30) through: $(-2,-5)$, slope $=2$

31) through: $(-4,2)$, slope $=-\frac{1}{2}$

32) through: $(5,3)$, slope $=\frac{6}{5}$

Write the point-slope form of the equation of the line through the given points.

33) through: $(-4, 3)$ and $(-3, 1)$

34) through: $(1, 3)$ and $(-3, 3)$

35) through: $(5, 1)$ and $(-3, 0)$

36) through: $(-4, 5)$ and $(4, 4)$

37) through: $(-4, -2)$ and $(0, 4)$

38) through: $(-4, 1)$ and $(4, 4)$

39) through: $(3, 5)$ and $(-5, 3)$

40) through: $(-1, -4)$ and $(-5, 0)$

41) through: $(3, -3)$ and $(-4, 5)$

42) through: $(-1, -5)$ and $(-5, -4)$

Write the slope-intercept form of the equation of the line through the given points.

43) through: $(-5, 1)$ and $(-1, -2)$

44) through: $(-5, -1)$ and $(5, -2)$

45) through: $(-5, 5)$ and $(2, -3)$

46) through: $(1, -1)$ and $(-5, -4)$

47) through: $(4, 1)$ and $(1, 4)$

48) through: $(0, 1)$ and $(-3, 0)$

49) through: $(0, 2)$ and $(5, -3)$

50) through: $(0, 2)$ and $(2, 4)$

51) through: $(0, 3)$ and $(-1, -1)$

52) through: $(-2, 0)$ and $(5, 3)$

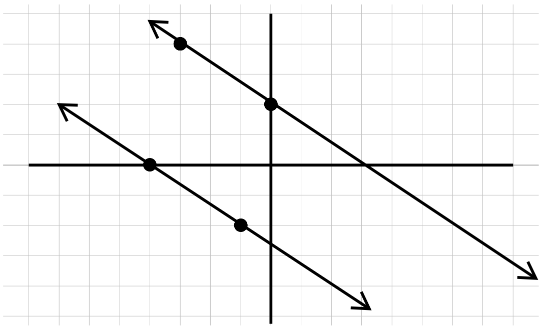
2.5

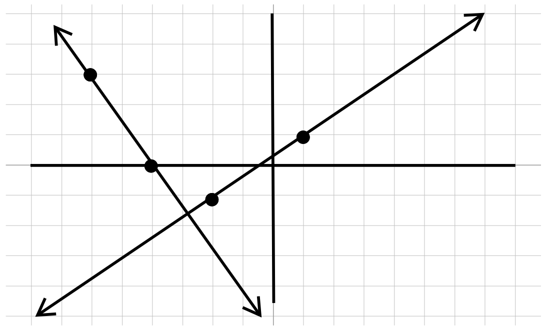
Graphing - Parallel and Perpendicular Lines

Objective: Identify the equation of a line given a parallel or perpendicular line.

There is an interesting connection between the slope of lines that are parallel and the slope of lines that are perpendicular (meet at a right angle). This is shown in the following example.

Example 145.





The above graph has two parallel lines. The slope of the top line is down 2, run 3, or $-\frac{2}{3}$. The slope of the bottom line is down 2, run 3 as well, or $-\frac{2}{3}$.

The above graph has two perpendicular lines. The slope of the flatter line is up 2, run 3 or $\frac{2}{3}$. The slope of the steeper line is down 3, run 2 or $-\frac{3}{2}$.

World View Note: Greek Mathematician Euclid lived around 300 BC and published a book titled, *The Elements*. In it is the famous parallel postulate which mathematicians have tried for years to drop from the list of postulates. The attempts have failed, yet all the work done has developed new types of geometries!

As the above graphs illustrate, parallel lines have the same slope and perpendicular lines have opposite (one positive, one negative) reciprocal (flipped fraction) slopes. We can use these properties to make conclusions about parallel and perpendicular lines.

Example 146.

Find the slope of a line parallel to $5y-2x=7$.

$$\begin{array}{rl}
5y-2x=7 & \text{To find the slope we will put equation in slope} - \text{intercept form} \\
\underline{+2x+2x} & \text{Add } 2x \text{ to both sides} \\
5y=2x+7 & \text{Put } x \text{ term first} \\
\overline{5}\quad\overline{5}\quad\overline{5} & \text{Divide each term by 5} \\
y=\dfrac{2}{5}x+\dfrac{7}{5} & \text{The slope is the coefficient of } x
\end{array}$$

$$m=\frac{2}{5} \quad \text{Slope of first line. Parallel lines have the same slope}$$

$$m=\frac{2}{5} \quad \text{Our Solution}$$

Example 147.

Find the slope of a line perpendicular to $3x-4y=2$

$$\begin{array}{rl}
3x-4y=2 & \text{To find slope we will put equation in slope} - \text{intercept form} \\
\underline{-3x \qquad -3x} & \text{Subtract } 3x \text{ from both sides} \\
-4y=-3x+2 & \text{Put } x \text{ term first} \\
\overline{-4} \quad \overline{-4}\ \overline{-4} & \text{Divide each term by } -4 \\
y=\frac{3}{4}x-\frac{1}{2} & \text{The slope is the coefficient of } x
\end{array}$$

$$m=\frac{3}{4} \quad \text{Slope of first lines. Perpendicular lines have opposite reciprocal slopes}$$

$$m=-\frac{4}{3} \quad \text{Our Solution}$$

Once we have a slope, it is possible to find the complete equation of the second line if we know one point on the second line.

Example 148.

Find the equation of a line through $(4,-5)$ and parallel to $2x-3y=6$.

$$\begin{array}{rl}
2x-3y=6 & \text{We first need slope of parallel line} \\
\underline{-2x \qquad -2x} & \text{Subtract } 2x \text{ from each side} \\
-3y=-2x+6 & \text{Put } x \text{ term first} \\
\overline{-3} \quad \overline{-3}\,\overline{-3} & \text{Divide each term by } -3 \\
y=\frac{2}{3}x-2 & \text{Identify the slope, the coefficient of } x
\end{array}$$

$$m=\frac{2}{3} \quad \text{Parallel lines have the same slope}$$

$$m=\frac{2}{3} \quad \text{We will use this slope and our point } (4,-5)$$

$$\begin{array}{rl}
y-y_1=m(x-x_1) & \text{Plug this information into point slope formula} \\
y-(-5)=\frac{2}{3}(x-4) & \text{Simplify signs}
\end{array}$$

$$y+5=\frac{2}{3}(x-4) \quad \text{Our Solution}$$

Example 149.

Find the equation of the line through $(6, -9)$ perpendicular to $y = -\frac{3}{5}x + 4$ in slope-intercept form.

$$
\begin{array}{rl}
y = -\dfrac{3}{5}x + 4 & \text{Identify the slope, coefficient of } x \\[2ex]
m = -\dfrac{3}{5} & \text{Perpendicular lines have opposite reciprocal slopes} \\[2ex]
m = \dfrac{5}{3} & \text{We will use this slope and our point } (6, -9) \\[2ex]
y - y_1 = m(x - x_1) & \text{Plug this information into point} - \text{slope formula} \\[2ex]
y - (-9) = \dfrac{5}{3}(x - 6) & \text{Simplify signs} \\[2ex]
y + 9 = \dfrac{5}{3}(x - 6) & \text{Distribute slope} \\[2ex]
y + 9 = \dfrac{5}{3}x - 10 & \text{Solve for } y \\
\underline{-9 \qquad\quad -9} & \text{Subtract 9 from both sides} \\
y = \dfrac{5}{3}x - 19 & \text{Our Solution}
\end{array}
$$

Zero slopes and no slopes may seem like opposites (one is a horizontal line, one is a vertical line). Because a horizontal line is perpendicular to a vertical line we can say that no slope and zero slope are actually perpendicular slopes!

Example 150.

Find the equation of the line through $(3, 4)$ perpendicular to $x = -2$

$$
\begin{array}{rl}
x = -2 & \text{This equation has no slope, } a \text{ vertical line} \\
\text{no slope} & \text{Perpendicular line then would have } a \text{ zero slope} \\
m = 0 & \text{Use this and our point } (3, 4) \\
y - y_1 = m(x - x_1) & \text{Plug this information into point} - \text{slope formula} \\
y - 4 = 0(x - 3) & \text{Distribute slope} \\
y - 4 = 0 & \text{Solve for } y \\
\underline{+4 + 4} & \text{Add 4 to each side} \\
y = 4 & \text{Our Solution}
\end{array}
$$

Being aware that to be perpendicular to a vertical line means we have a horizontal line through a y value of 4, thus we could have jumped from this point right to the solution, $y = 4$.

2.5 Practice - Parallel and Perpendicular Lines

Find the slope of a line parallel to each given line.

1) $y = 2x + 4$

2) $y = -\frac{2}{3}x + 5$

3) $y = 4x - 5$

4) $y = -\frac{10}{3}x - 5$

5) $x - y = 4$

6) $6x - 5y = 20$

7) $7x + y = -2$

8) $3x + 4y = -8$

Find the slope of a line perpendicular to each given line.

9) $x = 3$

10) $y = -\frac{1}{2}x - 1$

11) $y = -\frac{1}{3}x$

12) $y = \frac{4}{5}x$

13) $x - 3y = -6$

14) $3x - y = -3$

15) $x + 2y = 8$

16) $8x - 3y = -9$

Write the point-slope form of the equation of the line described.

17) through: $(2, 5)$, parallel to $x = 0$

18) through: $(5, 2)$, parallel to $y = \frac{7}{5}x + 4$

19) through: $(3, 4)$, parallel to $y = \frac{9}{2}x - 5$

20) through: $(1, -1)$, parallel to $y = -\frac{3}{4}x + 3$

21) through: $(2, 3)$, parallel to $y = \frac{7}{5}x + 4$

22) through: $(-1, 3)$, parallel to $y = -3x - 1$

23) through: $(4, 2)$, parallel to $x = 0$

24) through: $(1, 4)$, parallel to $y = \frac{7}{5}x + 2$

25) through: $(1, -5)$, perpendicular to $-x + y = 1$

26) through: $(1, -2)$, perpendicular to $-x + 2y = 2$

27) through: $(5, 2)$, perpendicular to $5x + y = -3$

28) through: $(1, 3)$, perpendicular to $-x + y = 1$

29) through: $(4, 2)$, perpendicular to $-4x + y = 0$

30) through: $(-3, -5)$, perpendicular to $3x + 7y = 0$

31) through: $(2, -2)$ perpendicular to $3y - x = 0$

32) through: $(-2, 5)$. perpendicular to $y - 2x = 0$

Write the slope-intercept form of the equation of the line described.

33) through: $(4, -3)$, parallel to $y = -2x$

34) through: $(-5, 2)$, parallel to $y = \frac{3}{5}x$

35) through: $(-3, 1)$, parallel to $y = -\frac{4}{3}x - 1$

36) through: $(-4, 0)$, parallel to $y = -\frac{5}{4}x + 4$

37) through: $(-4, -1)$, parallel to $y = -\frac{1}{2}x + 1$

38) through: $(2, 3)$, parallel to $y = \frac{5}{2}x - 1$

39) through: $(-2, -1)$, parallel to $y = -\frac{1}{2}x - 2$

40) through: $(-5, -4)$, parallel to $y = \frac{3}{5}x - 2$

41) through: $(4, 3)$, perpendicular to $x + y = -1$

42) through: $(-3, -5)$, perpendicular to $x + 2y = -4$

43) through: $(5, 2)$, perpendicular to $x = 0$

44) through: $(5, -1)$, perpendicular to $-5x + 2y = 10$

45) through: $(-2, 5)$, perpendicular to $-x + y = -2$

46) through: $(2, -3)$, perpendicular to $-2x + 5y = -10$

47) through: $(4, -3)$, perpendicular to $-x + 2y = -6$

48) through: $(-4, 1)$, perpendicular to $4x + 3y = -9$

Chapter 3 : Inequalities

3.1

Inequalities - Solve and Graph Inequalities

Objective: Solve, graph, and give interval notation for the solution to linear inequalities.

When we have an equation such as $x = 4$ we have a specific value for our variable. With inequalities we will give a range of values for our variable. To do this we will not use equals, but one of the following symbols:

$>$	Greater than
$\geqslant$	Greater than or equal to
$<$	Less than
$\leqslant$	Less than or equal to

World View Note: English mathematician Thomas Harriot first used the above symbols in 1631. However, they were not immediately accepted as symbols such as $\sqsubset$ and $\sqsupset$ were already coined by another English mathematician, William Oughtred.

If we have an expression such as $x < 4$, this means our variable can be any number smaller than 4 such as -2, 0, 3, 3.9 or even 3.999999999 as long as it is smaller than 4. If we have an expression such as $x \geqslant -2$, this means our variable can be any number greater than or equal to -2, such as $5, 0, -1, -1.9999$, or even -2.

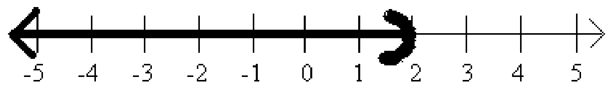
Because we don't have one set value for our variable, it is often useful to draw a picture of the solutions to the inequality on a number line. We will start from the value in the problem and bold the lower part of the number line if the variable is smaller than the number, and bold the upper part of the number line if the variable is larger. The value itself we will mark with brackets, either) or (for less than or greater than respectively, and] or [for less than or equal to or greater than or equal to respectively.

Once the graph is drawn we can quickly convert the graph into what is called interval notation. Interval notation gives two numbers, the first is the smallest value, the second is the largest value. If there is no largest value, we can use ∞ (infinity). If there is no smallest value, we can use $-\infty$ negative infinity. If we use either positive or negative infinity we will always use a curved bracket for that value.

Example 151.

Graph the inequality and give the interval notation

$x < 2$ Start at 2 and shade below
Use) for less than



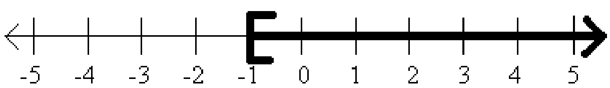
Our Graph

$(-\infty, 2)$ Interval Notation

Example 152.

Graph the inequality and give the interval notation

$y \geqslant -1$ Start at -1 and shade above
Use [for greater than or equal

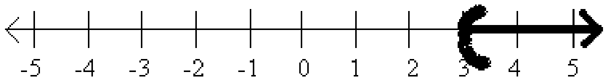


Our Graph

$[-1, \infty)$ Interval Notation

We can also take a graph and find the inequality for it.

Example 153.

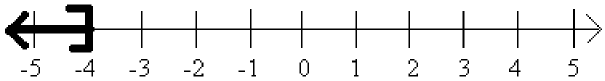


Give the inequality for the graph:

Graph starts at 3 and goes up or greater. Curved bracket means just greater than

$$x > 3 \quad \text{Our Solution}$$

Example 154.



Give the inequality for the graph:

Graph starts at -4 and goes down or less. Square bracket means less than or equal to

$$x \leqslant -4 \quad \text{Our Solution}$$

Generally when we are graphing and giving interval notation for an inequality we will have to first solve the inequality for our variable. Solving inequalities is very similar to solving equations with one exception. Consider the following inequality and what happens when various operations are done to it. Notice what happens to the inequality sign as we add, subtract, multiply and divide by both positive and negative numbers to keep the statment a true statement.

$$\begin{aligned}
5 > 1 &\quad \text{Add 3 to both sides} \\
8 > 4 &\quad \text{Subtract 2 from both sides} \\
6 > 2 &\quad \text{Multiply both sides by 3} \\
12 > 6 &\quad \text{Divide both sides by 2} \\
6 > 3 &\quad \text{Add } -1 \text{ to both sides} \\
5 > 2 &\quad \text{Subtract } -4 \text{ from both sides} \\
9 > 6 &\quad \text{Multiply both sides by } -2 \\
-18 < -12 &\quad \text{Divide both sides by } -6 \\
3 > 2 &\quad \text{Symbol flipped when we multiply or divide by } a \text{ negative!}
\end{aligned}$$

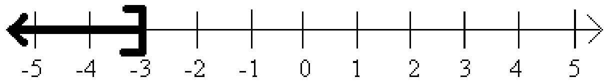
As the above problem illustrates, we can add, subtract, multiply, or divide on both sides of the inequality. But if we multiply or divide by a negative number, the symbol will need to flip directions. We will keep that in mind as we solve inequalities.

Example 155.

Solve and give interval notation

$$5 - 2x \geqslant 11 \quad \text{Subtract 5 from both sides}$$

$$\begin{array}{rl}
\underline{-5\qquad -5} & \\
-2x\geqslant 6 & \text{Divide both sides by } -2\\
\overline{-2}\ \overline{-2} & \text{Divide by } a \text{ negative} - \text{flip symbol!}\\
x\leqslant -3 & \text{Graph, starting at } -3\text{, going down with] for less than or equal to}
\end{array}$$



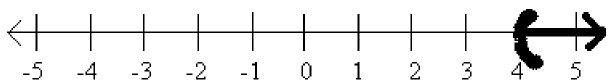
$$\begin{array}{rl}
(-\infty,-3] & \text{Interval Notation}
\end{array}$$

The inequality we solve can get as complex as the linear equations we solved. We will use all the same patterns to solve these inequalities as we did for solving equations. Just remember that any time we multiply or divide by a negative the symbol switches directions (multiplying or dividing by a positive does not change the symbol!)

Example 156.

Solve and give interval notation

$$\begin{array}{rl}
3(2x-4)+4x<4(3x-7)+8 & \text{Distribute}\\
6x-12+4x<12x-28+8 & \text{Combine like terms}\\
10x-12<12x-20 & \text{Move variable to one side}\\
\underline{-10x\qquad -10x} & \text{Subtract } 10x \text{ from both sides}\\
-12<2x-20 & \text{Add 20 to both sides}\\
\underline{+20\qquad +20} & \\
8<2x & \text{Divide both sides by 2}\\
\overline{2}\ \ \overline{2} & \\
4<x & \text{Be careful with graph, } x \text{ is larger!}
\end{array}$$



$$\begin{array}{rl}
(4,\infty) & \text{Interval Notation}
\end{array}$$

It is important to be careful when the inequality is written backwards as in the previous example ($4<x$ rather than $x>4$). Often students draw their graphs the wrong way when this is the case. The inequality symbol opens to the variable, this means the variable is greater than 4. So we must shade above the 4.

3.1 Practice - Solve and Graph Inequalities

Draw a graph for each inequality and give interval notation.

1) $n > -5$

2) $n > 4$

3) $-2 \geqslant k$

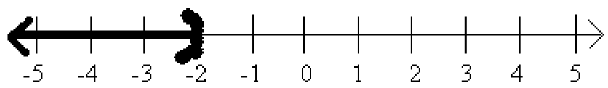
4) $1 \geqslant k$

5) $5 \geqslant x$

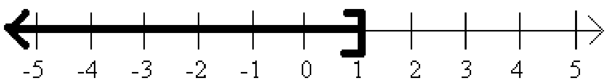
6) $-5 < x$

Write an inequality for each graph.

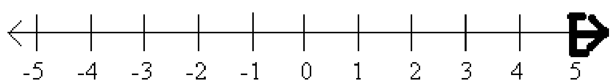
7)



8)



9)



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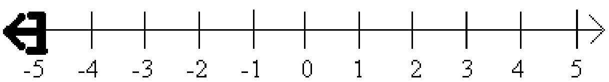
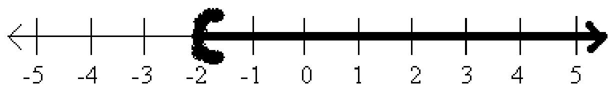
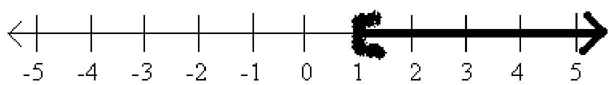


11)



12)



Solve each inequality, graph each solution, and give interval notation.

13) $\frac{x}{11} \geqslant 10$

14) $-2 \leqslant \frac{n}{13}$

15) $2 + r < 3$

16) $\frac{m}{5} \leqslant -\frac{6}{5}$

17) $8 + \frac{n}{3} \geqslant 6$

18) $11 > 8 + \frac{x}{2}$

19) $2 > \frac{a-2}{5}$

20) $\frac{v-9}{-4} \leqslant 2$

21) $-47 \geqslant 8 - 5x$

22) $\frac{6+x}{12} \leqslant -1$

23) $-2(3+k) < -44$

24) $-7n - 10 \geqslant 60$

25) $18 < -2(-8+p)$

26) $5 \geqslant \frac{x}{5} + 1$

27) $24 \geqslant -6(m-6)$

28) $-8(n-5) \geqslant 0$

29) $-r - 5(r-6) < -18$

30) $-60 \geqslant -4(-6x-3)$

31) $24 + 4b < 4(1+6b)$

32) $-8(2-2n) \geqslant -16 + n$

33) $-5v - 5 < -5(4v+1)$

34) $-36 + 6x > -8(x+2) + 4x$

35) $4 + 2(a+5) < -2(-a-4)$

36) $3(n+3) + 7(8-8n) < 5n + 5 + 2$

37) $-(k-2) > -k - 20$

38) $-(4-5p) + 3 \geqslant -2(8-5p)$

3.2

Inequalities - Compound Inequalities

Objective: Solve, graph and give interval notation to the solution of compound inequalities.

Several inequalities can be combined together to form what are called compound inequalities. There are three types of compound inequalities which we will investigate in this lesson.

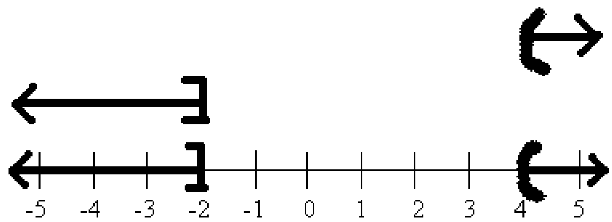
The first type of a compound inequality is an OR inequality. For this type of inequality we want a true statment from either one inequality OR the other inequality OR both. When we are graphing these type of inequalities we will graph each individual inequality above the number line, then move them both down together onto the actual number line for our graph that combines them together.

When we give interval notation for our solution, if there are two different parts to the graph we will put a $\cup$ (union) symbol between two sets of interval notation, one for each part.

Example 157.

Solve each inequality, graph the solution, and give interval notation of solution

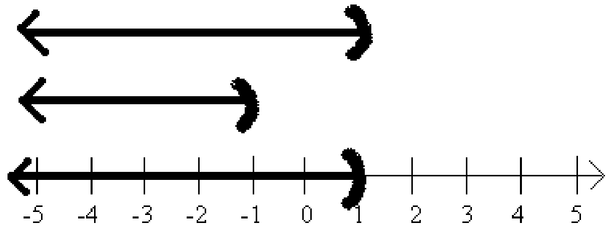
$$\begin{array}{rl} 2x-5>3 \text{ or } 4-x\geqslant 6 & \text{Solve each inequality} \\ \underline{+5+5} \quad \underline{-4 \qquad -4} & \text{Add or subtract first} \\ 2x>8 \text{ or } -x\geqslant 2 & \text{Divide} \\ \overline{2}\ \ \overline{2} \qquad \overline{-1}\ \overline{-1} & \text{Dividing by negative flips sign} \\ x>4 \text{ or } x\leqslant -2 & \text{Graph the inequalities separatly above number line} \end{array}$$

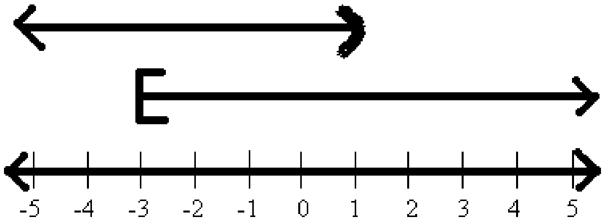


$$(-\infty,-2]\cup(4,\infty) \quad \text{Interval Notation}$$

World View Note: The symbol for infinity was first used by the Romans, although at the time the number was used for 1000. The greeks also used the symbol for 10,000.

There are several different results that could result from an OR statement. The graphs could be pointing different directions, as in the graph above, or pointing in the same direction as in the graph below on the left, or pointing opposite directions, but overlapping as in the graph below on the right. Notice how interval notation works for each of these cases.





As the graphs overlap, we take the largest graph for our solution.

Interval Notation: $(-\infty, 1)$

When the graphs are combined they cover the entire number line.

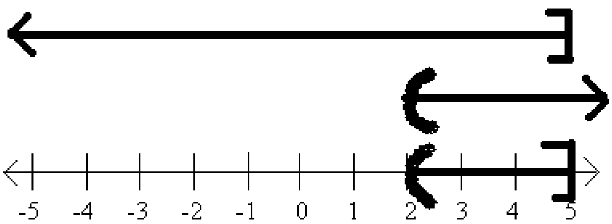
Interval Notation: $(-\infty, \infty)$ or $\mathbb{R}$

The second type of compound inequality is an AND inequality. AND inequalities require both statements to be true. If one is false, they both are false. When we graph these inequalities we can follow a similar process, first graph both inequalities above the number line, but this time only where they overlap will be drawn onto the number line for our final graph. When our solution is given in interval notation it will be expressed in a manner very similar to single inequalities (there is a symbol that can be used for AND, the intersection - $\cap$, but we will not use it here).

Example 158.

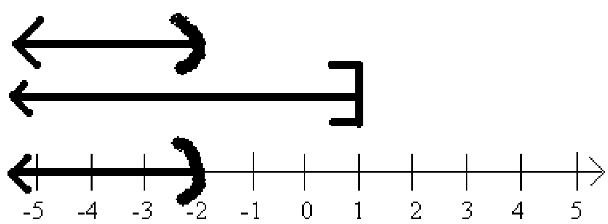
Solve each inequality, graph the solution, and express it interval notation.

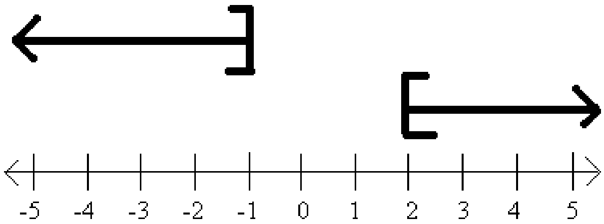
$$\begin{array}{rcl l}
2x+8\geqslant 5x-7 & \text{and} & 5x-3>3x+1 & \text{Move variables to one side} \\
\underline{-2x \qquad -2x} & & \underline{-3x \qquad -3x} & \\
8\geqslant 3x-7 & \text{and} & 2x-3>1 & \text{Add 7 or 3 to both sides} \\
\underline{+7 \qquad +7} & & \underline{+3+3} & \\
15\geqslant 3x & \text{and} & 2x>4 & \text{Divide} \\
\overline{3} \quad \overline{3} & & \overline{2} \quad \overline{2} & \\
5\geqslant x & \text{and} & x>2 & \text{Graph, } x \text{ is smaller (or equal) than 5, greater than 2}
\end{array}$$



$(2,5]$ Interval Notation

Again, as we graph AND inequalities, only the overlapping parts of the individual graphs makes it to the final number line. As we graph AND inequalities there are also three different types of results we could get. The first is shown in the above example. The second is if the arrows both point the same way, this is shown below on the left. The third is if the arrows point opposite ways but don't overlap, this is shown below on the right. Notice how interval notation is expressed in each case.





In this graph, the overlap is only the smaller graph, so this is what makes it to the final number line.

Interval Notation: $(-\infty, -2)$

In this graph there is no overlap of the parts. Because their is no overlap, no values make it to the final number line.

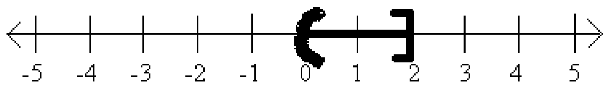
Interval Notation: No Solution or $\varnothing$

The third type of compound inequality is a special type of AND inequality. When our variable (or expression containing the variable) is between two numbers, we can write it as a single math sentence with three parts, such as $5 < x \leqslant 8$, to show x is between 5 and 8 (or equal to 8). When solving these type of inequalities, because there are three parts to work with, to stay balanced we will do the same thing to all three parts (rather than just both sides) to isolate the variable in the middle. The graph then is simply the values between the numbers with appropriate brackets on the ends.

Example 159.

Solve the inequality, graph the solution, and give interval notation.

$$\begin{array}{rl}
-6 \leqslant -4x+2 < 2 & \text{Subtract 2 from all three parts} \\
\underline{-2 \qquad\quad -2 -2} & \\
-8 \leqslant -4x < 0 & \text{Divide all three parts by } -4 \\
\overline{-4} \quad \overline{-4}\ \overline{-4} & \text{Dividing by } a \text{ negative flips the symbols} \\
2 \geqslant x > 0 & \text{Flip entire statement so values get larger left to right} \\
0 < x \leqslant 2 & \text{Graph } x \text{ between 0 and 2}
\end{array}$$



$(0, 2]$ Interval Notation

3.2 Practice - Compound Inequalities

Solve each compound inequality, graph its solution, and give interval notation.

1) $\frac{n}{3} \leqslant -3 \text{ or } -5n \leqslant -10$

2) $6m \geqslant -24 \text{ or } m - 7 < -12$

3) $x + 7 \geqslant 12 \text{ or } 9x < -45$

4) $10r > 0 \text{ or } r - 5 < -12$

5) $x - 6 < -13 \text{ or } 6x \leqslant -60$

6) $9 + n < 2 \text{ or } 5n > 40$

7) $\frac{v}{8} > -1 \text{ and } v - 2 < 1$

8) $-9x < 63 \text{ and } \frac{x}{4} < 1$

9) $-8 + b < -3 \text{ and } 4b < 20$

10) $-6n \leqslant 12 \text{ and } \frac{n}{3} \leqslant 2$

11) $a + 10 \geqslant 3 \text{ and } 8a \leqslant 48$

12) $-6 + v \geqslant 0 \text{ and } 2v > 4$

13) $3 \leqslant 9 + x \leqslant 7$

14) $0 \geqslant \frac{x}{9} \geqslant -1$

15) $11 < 8 + k \leqslant 12$

16) $-11 \leqslant n - 9 \leqslant -5$

17) $-3 < x - 1 < 1$

18) $1 \leqslant \frac{p}{8} \leqslant 0$

19) $-4 < 8 - 3m \leqslant 11$

20) $3 + 7r > 59 \text{ or } -6r - 3 > 33$

21) $-16 \leqslant 2n - 10 \leqslant -22$

22) $-6 - 8x \geqslant -6 \text{ or } 2 + 10x > 82$

23) $-5b + 10 \leqslant 30 \text{ and } 7b + 2 \leqslant -40$

24) $n + 10 \geqslant 15 \text{ or } 4n - 5 < -1$

25) $3x - 9 < 2x + 10 \text{ and } 5 + 7x \leqslant 10x - 10$

26) $4n + 8 < 3n - 6 \text{ or } 10n - 8 \geqslant 9 + 9n$

27) $-8 - 6v \leqslant 8 - 8v \text{ and } 7v + 9 \leqslant 6 + 10v$

28) $5 - 2a \geqslant 2a + 1 \text{ or } 10a - 10 \geqslant 9a + 9$

29) $1 + 5k \leqslant 7k - 3 \text{ or } k - 10 > 2k + 10$

30) $8 - 10r \leqslant 8 + 4r \text{ or } -6 + 8r < 2 + 8r$

31) $2x + 9 \geqslant 10x + 1 \text{ and } 3x - 2 < 7x + 2$

32) $-9m + 2 < -10 - 6m \text{ or } -m + 5 \geqslant 10 + 4m$

3.3

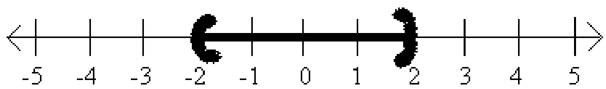
Inequalities - Absolute Value Inequalities

Objective: Solve, graph and give interval notation for the solution to inequalities with absolute values.

When an inequality has an absolute value we will have to remove the absolute value in order to graph the solution or give interval notation. The way we remove the absolute value depends on the direction of the inequality symbol.

Consider $|x| < 2$.

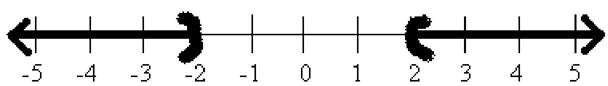
Absolute value is defined as distance from zero. Another way to read this inequality would be the distance from zero is less than 2. So on a number line we will shade all points that are less than 2 units away from zero.



This graph looks just like the graphs of the three part compound inequalities! When the absolute value is **less than** a number we will remove the absolute value by changing the problem to a three part inequality, with the negative value on the left and the positive value on the right. So $|x| < 2$ becomes $-2 < x < 2$, as the graph above illustrates.

Consider $|x| > 2$.

Absolute value is defined as distance from zero. Another way to read this inequality would be the distance from zero is greater than 2. So on the number line we shade all points that are more than 2 units away from zero.



This graph looks just like the graphs of the OR compound inequalities! When the absolute value is **greater than** a number we will remove the absolute value by changing the problem to an OR inequality, the first inequality looking just like the problem with no absolute value, the second flipping the inequality symbol and changing the value to a negative. So $|x| > 2$ becomes $x > 2$ or $x < -2$, as the graph above illustrates.

World View Note: The phrase "absolute value" comes from German mathematician Karl Weierstrass in 1876, though he used the absolute value symbol for complex numbers. The first known use of the symbol for integers comes from a 1939 edition of a college algebra text!

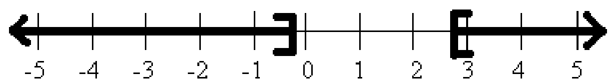
For all absolute value inequalities we can also express our answers in interval notation which is done the same way it is done for standard compound inequalities.

We can solve absolute value inequalities much like we solved absolute value equations. Our first step will be to isolate the absolute value. Next we will remove the absolute value by making a three part inequality if the absolute value is less than a number, or making an OR inequality if the absolute value is greater than a number. Then we will solve these inequalites. Remember, if we multiply or divide by a negative the inequality symbol will switch directions!

Example 160.

Solve, graph, and give interval notation for the solution

$$\begin{array}{rl}
|4x-5|\geqslant 6 & \text{Absolute value is greater, use OR}\\
4x-5\geqslant 6 \ \text{ OR } \ 4x-5\leqslant -6 & \text{Solve}\\
\underline{+5+5} \qquad\quad \underline{+5 \quad +5} & \text{Add 5 to both sides}\\
4x\geqslant 11 \ \text{ OR } \ 4x\leqslant -1 & \text{Divide both sides by 4}\\
\overline{4} \quad \overline{4} \qquad\quad \overline{4} \quad \overline{4} & \\
x\geqslant \dfrac{11}{4} \ \text{ OR } \ x\leqslant -\dfrac{1}{4} & \text{Graph}
\end{array}$$



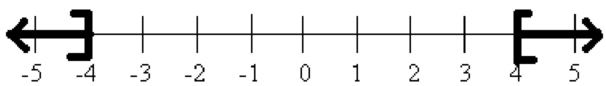
$$\left(-\infty,-\frac{1}{4}\right]\cup\left[\frac{11}{4},\infty\right) \qquad \text{Interval notation}$$

Example 161.

Solve, graph, and give interval notation for the solution

$$\begin{array}{rl}
-4-3|x|\leqslant -16 & \text{Add 4 to both sides}\\
\underline{+4 \qquad\qquad +4} & \\
-3|x|\leqslant -12 & \text{Divide both sides by } -3\\
\overline{-3} \qquad \overline{-3} & \text{Dividing by } a \text{ negative switches the symbol}\\
|x|\geqslant 4 & \text{Absolute value is greater, use OR}
\end{array}$$

$x \geqslant 4$ OR $x \leqslant -4$ Graph



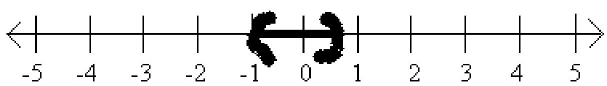
$$(-\infty, -4] \cup [4, \infty) \quad \text{Interval Notation}$$

In the previous example, we cannot combine -4 and -3 because they are not like terms, the -3 has an absolute value attached. So we must first clear the -4 by adding 4, then divide by -3. The next example is similar.

Example 162.

Solve, graph, and give interval notation for the solution

$$\begin{array}{rl}
9 - 2|4x + 1| > 3 & \text{Subtract 9 from both sides} \\
\underline{-9 \qquad\qquad -9} & \\
-2|4x + 1| > -6 & \text{Divide both sides by } -2 \\
\overline{-2} \qquad \overline{-2} & \text{Dividing by negative switches the symbol} \\
|4x + 1| < 3 & \text{Absolute value is less, use three part} \\
-3 < 4x + 1 < 3 & \text{Solve} \\
\underline{-1 \qquad -1 - 1} & \text{Subtract 1 from all three parts} \\
-4 < 4x < 2 & \text{Divide all three parts by 4} \\
\overline{4} \quad \overline{4} \quad \overline{4} & \\
-1 < x < \dfrac{1}{2} & \text{Graph}
\end{array}$$



$$\left(-1, \frac{1}{2}\right) \quad \text{Interval Notation}$$

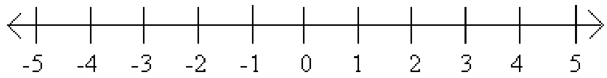
In the previous example, we cannot distribute the -2 into the absolute value. We can never distribute or combine things outside the absolute value with what is inside the absolute value. Our only way to solve is to first isolate the absolute value by clearing the values around it, then either make a compound inequality (and OR or a three part) to solve.

It is important to remember as we are solving these equations, the absolute value is always positive. If we end up with an absolute value is less than a negative number, then we will have no solution because absolute value will always be positive, greater than a negative. Similarly, if absolute value is greater than a negative, this will always happen. Here the answer will be all real numbers.

Example 163.

Solve, graph, and give interval notation for the solution

$$\begin{array}{rl}
12+4|6x-1|<4 & \text{Subtract 12 from both sides}\\
\underline{-12 \qquad\qquad -12} & \\
4|6x-1|<-8 & \text{Divide both sides by 4}\\
\overline{\quad 4 \quad}\qquad \overline{4} & \\
|6x-1|<-2 & \text{Absolute value can't be less than a negative}
\end{array}$$

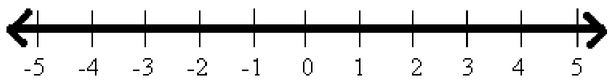


No Solution or $\varnothing$

Example 164.

Solve, graph, and give interval notation for the solution

$$\begin{array}{rl}
5-6|x+7|\leqslant 17 & \text{Subtract 5 from both sides}\\
\underline{-5 \qquad\qquad -5} & \\
-6|x+7|\leqslant 12 & \text{Divide both sides by } -6\\
\overline{\quad -6 \quad}\qquad \overline{-6} & \text{Dividing by a negative flips the symbol}\\
|x+7|\geqslant -2 & \text{Absolute value always greater than negative}
\end{array}$$



All Real Numbers or $\mathbb{R}$

3.3 Practice - Absolute Value Inequalities

Solve each inequality, graph its solution, and give interval notation.

1) $|x| < 3$

2) $|x| \leqslant 8$

3) $|2x| < 6$

4) $|x + 3| < 4$

5) $|x - 2| < 6$

6) $x - 8| < 12$

7) $|x - 7| < 3$

8) $|x + 3| \leqslant 4$

9) $|3x - 2| < 9$

10) $|2x + 5| < 9$

11) $1 + 2|x - 1| \leqslant 9$

12) $10 - 3|x - 2| \geqslant 4$

13) $6 - |2x - 5| > = 3$

14) $|x| > 5$

15) $|3x| > 5$

16) $|x - 4| > 5$

17) $|x = 3| > = 3$

18) $|2x - 4| > 6$

19) $|3x - 5| > \geqslant 3$

20) $3 - |2 - x| < 1$

21) $4 + 3|x - 1| > = 10$

22) $3 - 2|3x - 1| \geqslant - 7$

23) $3 - 2|x - 5| \leqslant - 15$

24) $4 - 6| - 6 - 3x| \leqslant - 5$

25) $- 2 - 3|4 - 2x| \geqslant - 8$

26) $- 3 - 2|4x - 5| \geqslant 1$

27) $4 - 5| - 2x - 7| < - 1$

28) $- 2 + 3|5 - x| \leqslant 4$

29) $3 - 2|4x - 5| \geqslant 1$

30) $- 2 - 3| - 3x - 5 \geqslant - 5$

31) $- 5 - 2|3x - 6| < - 8$

32) $6 - 3|1 - 4x| < - 3$

33) $4 - 4| - 2x + 6| > - 4$

34) $- 3 - 4| - 2x - 5| \geqslant - 7$

35) $| - 10 + x| \geqslant 8$

Chapter 4 : Systems of Equations

4.1

Systems of Equations - Graphing

Objective: Solve systems of equations by graphing and identifying the point of intersection.

We have solved problems like $3x - 4 = 11$ by adding 4 to both sides and then dividing by 3 (solution is $x = 5$). We also have methods to solve equations with more than one variable in them. It turns out that to solve for more than one variable we will need the same number of equations as variables. For example, to solve for two variables such as x and y we will need two equations. When we have several equations we are using to solve, we call the equations a **system of equations**. When solving a system of equations we are looking for a solution that works in both equations. This solution is usually given as an ordered pair (x, y). The following example illustrates a solution working in both equations

Example 165.

Show (2,1) is the solution to the system $\begin{array}{l} 3x - y = 5 \\ x + y = 3 \end{array}$

$$\begin{array}{rl} (2,1) & \text{Identify } x \text{ and } y \text{ from the orderd pair} \\ x = 2, y = 1 & \text{Plug these values into each equation} \\ & \\ 3(2) - (1) = 5 & \text{First equation} \\ 6 - 1 = 5 & \text{Evaluate} \\ 5 = 5 & \text{True} \\ & \\ (2) + (1) = 3 & \text{Second equation, evaluate} \\ 3 = 3 & \text{True} \end{array}$$

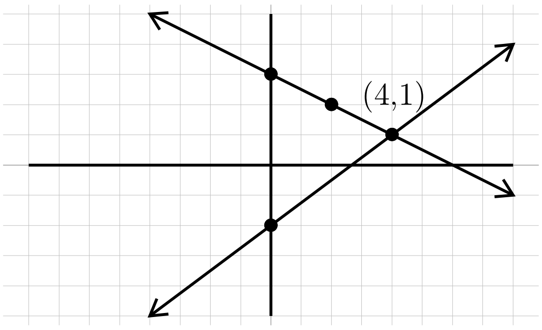
As we found a true statement for both equations we know (2,1) is the solution to the system. It is in fact the only combination of numbers that works in both equations. In this lesson we will be working to find this point given the equations. It seems to follow that if we use points to describe the solution, we can use graphs to find the solutions.

If the graph of a line is a picture of all the solutions, we can graph two lines on the same coordinate plane to see the solutions of both equations. We are interested in the point that is a solution for both lines, this would be where the lines intersect! If we can find the intersection of the lines we have found the solution that works in both equations.

Example 166.

$$\begin{aligned} y &= -\frac{1}{2}x+3 \\ y &= \frac{3}{4}x-2 \end{aligned} \quad \text{To graph we identify slopes and } y-\text{intercepts}$$

$$\begin{aligned} \text{First: } m &= -\frac{1}{2}, b=3 \\ \text{Second: } m &= \frac{3}{4}, b=-2 \end{aligned} \quad \text{Now we can graph both lines on the same plane.}$$



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Remember a negative slope is downhill!

Find the intersection point, (4,1)

(4,1) Our Solution

Often our equations won't be in slope-intercept form and we will have to solve both equations for y first so we can idenfity the slope and y-intercept.

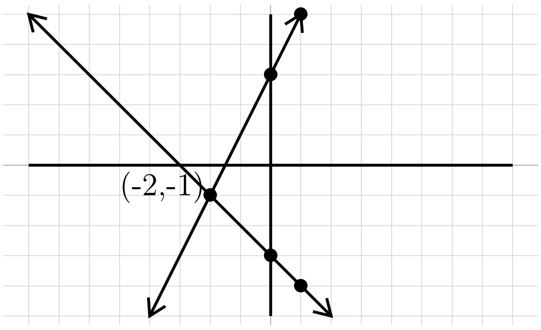
Example 167.

$$\begin{aligned} 6x-3y &= -9 \\ 2x+2y &= -6 \end{aligned} \quad \text{Solve each equation for } y$$

$$\begin{array}{ccl}
6x-3y=-9 & 2x+2y=-6 & \\
\underline{-6x \qquad -6x} & \underline{-2x \qquad -2x} & \text{Subtract } x \text{ terms} \\
-3y=-6x-9 & 2y=-2x-6 & \text{Put } x \text{ terms first} \\
\overline{-3} \quad \overline{-3} \; \overline{-3} & \overline{2} \quad \overline{2} \; \overline{2} & \text{Divide by coefficient of } y \\
y=2x+3 & y=-x-3 & \text{Identify slope and } y-\text{intercepts}
\end{array}$$

$$\text{First: } m = \frac{2}{1}, b = 3$$
$$\text{Second: } m = -\frac{1}{1}, b = -3$$

Now we can graph both lines on the same plane



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Remember a negative slope is downhill!

Find the intersection point, $(-2,-1)$

$(-2,-1)$ Our Solution

As we are graphing our lines, it is possible to have one of two unexpected results. These are shown and discussed in the next two example.

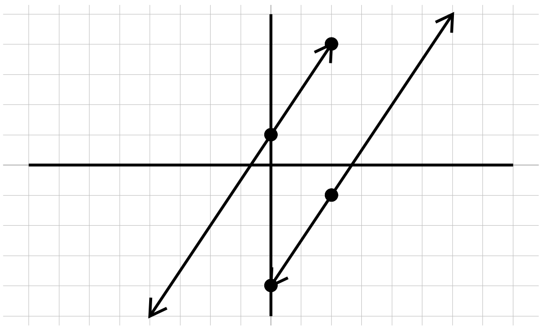
Example 168.

$$y = \frac{3}{2}x - 4$$
$$y = \frac{3}{2}x + 1$$

Identify slope and $y-$intercept of each equation

$$\text{First: } m = \frac{3}{2}, b = -4$$
$$\text{Second: } m = \frac{3}{2}, b = 1$$

Now we can graph both equations on the same plane



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

The two lines do not intersect! They are parallel! If the lines do not intersect we know that there is no point that works in both equations, there is no solution

$\varnothing$ No Solution

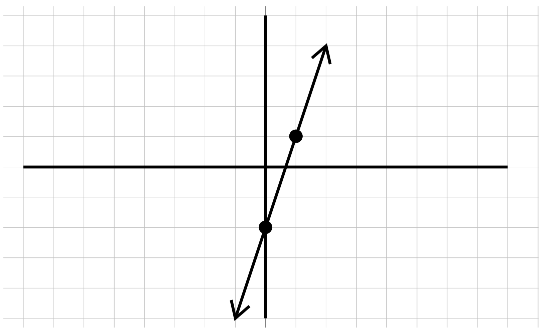
We also could have noticed that both lines had the same slope. Remembering that parallel lines have the same slope we would have known there was no solution even without having to graph the lines.

Example 169.

$$\begin{aligned} 2x - 6y &= 12 \\ 3x - 9y &= 18 \end{aligned} \quad \text{Solve each equation for } y$$

$$\begin{array}{rcrl}
2x - 6y = 12 & & 3x - 9y = 18 & \\
\underline{-2x \qquad -2x} & & \underline{-3x \qquad -3x} & \text{Subtract } x \text{ terms} \\
-6y = -2x + 12 & & -9y = -3x + 18 & \text{Put } x \text{ terms first} \\
\overline{-6} \quad \overline{-6} \quad \overline{-6} & & \overline{-9} \quad \overline{-9} \quad \overline{-9} & \text{Divide by coefficient of } y \\
y = \frac{1}{3}x - 2 & & y = \frac{1}{3}x - 2 & \text{Identify the slopes and } y-\text{intercepts}
\end{array}$$

$$\begin{aligned} \text{First: } m &= \frac{1}{3}, b = -2 \\ \text{Second: } m &= \frac{1}{3}, b = -2 \end{aligned} \quad \text{Now we can graph both equations together}$$



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Both equations are the same line! As one line is directly on top of the other line, we can say that the lines "intersect" at all the points! Here we say we have infinite solutions

Once we had both equations in slope-intercept form we could have noticed that the equations were the same. At this point we could have stated that there are infinite solutions without having to go through the work of graphing the equations.

World View Note: The Babylonians were the first to work with systems of equations with two variables. However, their work with systems was quickly passed by the Greeks who would solve systems of equations with three or four variables and around 300 AD, developed methods for solving systems with any number of unknowns!

4.1 Practice - Graphing

Solve each equation by graphing.

1) $y = -x + 1$
$y = -5x - 3$

2) $y = -\frac{5}{4}x - 2$
$y = -\frac{1}{4}x + 2$

3) $y = -3$
$y = -x - 4$

4) $y = -x - 2$
$y = \frac{2}{3}x + 3$

5) $y = -\frac{3}{4}x + 1$
$y = -\frac{3}{4}x + 2$

6) $y = 2x + 2$
$y = -x - 4$

7) $y = \frac{1}{3}x + 2$
$y = -\frac{5}{3}x - 4$

8) $y = 2x - 4$
$y = \frac{1}{2}x + 2$

9) $y = \frac{5}{3}x + 4$
$y = -\frac{2}{3}x - 3$

10) $y = \frac{1}{2}x + 4$
$y = \frac{1}{2}x + 1$

11) $x + 3y = -9$
$5x + 3y = 3$

12) $x + 4y = -12$
$2x + y = 4$

13) $x - y = 4$
$2x + y = -1$

14) $6x + y = -3$
$x + y = 2$

15) $2x + 3y = -6$
$2x + y = 2$

16) $3x + 2y = 2$
$3x + 2y = -6$

17) $2x + y = 2$
$x - y = 4$

18) $x + 2y = 6$
$5x - 4y = 16$

19) $2x + y = -2$
$x + 3y = 9$

20) $x - y = 3$
$5x + 2y = 8$

21) $0 = -6x - 9y + 36$
$12 = 6x - 3y$

22) $-2y + x = 4$
$2 = -x + \frac{1}{2}y$

23) $2x - y = -1$
$0 = -2x - y - 3$

24) $-2y = -4 - x$
$-2y = -5x + 4$

25) $3 + y = -x$
$-4 - 6x = -y$

26) $16 = -x - 4y$
$-2x = -4 - 4y$

27) $-y + 7x = 4$
$-y - 3 + 7x = 0$

28) $-4 + y = x$
$x + 2 = -y$

29) $-12 + x = 4y$
$12 - 5x = 4y$

30) $-5x + 1 = -y$
$-y + x = -3$

4.2

Systems of Equations - Substitution

Objective: Solve systems of equations using substitution.

When solving a system by graphing has several limitations. First, it requires the graph to be perfectly drawn, if the lines are not straight we may arrive at the wrong answer. Second, graphing is not a great method to use if the answer is really large, over 100 for example, or if the answer is a decimal the that graph will not help us find, 3.2134 for example. For these reasons we will rarely use graphing to solve our systems. Instead, an algebraic approach will be used.

The first algebraic approach is called substitution. We will build the concepts of substitution through several example, then end with a five-step process to solve problems using this method.

Example 170.

$x=5$ $y=2x-3$	We already know $x=5$, substitute this into the other equation
$y=2(\mathbf{5})-3$	Evaluate, multiply first
$y=10-3$	Subtract
$y=7$	We now also have y
$(5,7)$	Our Solution

When we know what one variable equals we can plug that value (or expression) in for the variable in the other equation. It is very important that when we substitute, the substituted value goes in parenthesis. The reason for this is shown in the next example.

Example 171.

$2x-3y=7$ $y=3x-7$	We know $y=3x-7$, substitute this into the other equation
$2x-3(\boldsymbol{3x-7})=7$	Solve this equation, distributing -3 first

$$\begin{array}{rl}
2x-9x+21=7 & \text{Combine like terms } 2x-9x \\
-7x+21=7 & \text{Subtract } 21 \\
\underline{\mathbf{-21\ -21}} & \\
-7x=-14 & \text{Divide by } -7 \\
\overline{\mathbf{-7}}\quad \overline{\mathbf{-7}} & \\
x=2 & \text{We now have our } x\text{, plug into the } y= \text{ equation to find } y \\
y=3(\mathbf{2})-7 & \text{Evaluate, multiply first} \\
y=6-7 & \text{Subtract} \\
y=-1 & \text{We now also have } y \\
(2,-1) & \text{Our Solution}
\end{array}$$

By using the entire expression $3x-7$ to replace y in the other equation we were able to reduce the system to a single linear equation which we can easily solve for our first variable. However, the lone variable (a variable without a coefficient) is not always alone on one side of the equation. If this happens we can isolate it by solving for the lone variable.

Example 172.

$$\begin{array}{rl}
3x+2y=1 & \\
\boldsymbol{x}-5y=6 & \text{Lone variable is } x\text{, isolate by adding } 5y \text{ to both sides.} \\
\underline{\mathbf{+5y+5y}} & \\
x=6+5y & \text{Substitute this into the untouched equation} \\
3(\mathbf{6+5y})+2y=1 & \text{Solve this equation, distributing 3 first} \\
18+15y+2y=1 & \text{Combine like terms } 15y+2y \\
18+17y=1 & \text{Subtract 18 from both sides} \\
\underline{\mathbf{-18\qquad -18}} & \\
17y=-17 & \text{Divide both sides by 17} \\
\overline{\mathbf{17}}\quad \overline{\mathbf{17}} & \\
y=-1 & \text{We have our } y\text{, plug this into the } x= \text{ equation to find } x \\
x=6+5(\mathbf{-1}) & \text{Evaluate, multiply first} \\
x=6-5 & \text{Subtract} \\
x=1 & \text{We now also have } x \\
(1,-1) & \text{Our Solution}
\end{array}$$

The process in the previous example is how we will solve problems using substitution. This process is described and illustrated in the following table which lists the five steps to solving by substitution.

Problem	$4x-2y=2$ $2x+y=-5$
1. Find the lone variable	Second Equation, y $2x+\boldsymbol{y}=-5$
2. Solve for the lone variable	$-2x \quad -2x$ $\boldsymbol{y=-5-2x}$
3. Substitute into the untouched equation	$4x-2(\boldsymbol{-5-2x})=2$
4. Solve	$4x+10+4x=2$ $8x+10=2$ $\underline{-10-10}$ $\frac{8x}{8}=\frac{-8}{8}$ $\boldsymbol{x=-1}$
5. Plug into lone variable equation and evaluate	$y=-5-2(\boldsymbol{-1})$ $y=-5+2$ $\boldsymbol{y=-3}$
Solution	$(-1,-3)$

Sometimes we have several lone variables in a problem. In this case we will have the choice on which lone variable we wish to solve for, either will give the same final result.

Example 173.

$$\begin{array}{rl}
\boldsymbol{x}+y=5 & \text{Find the lone variable: } x \text{ or } y \text{ in first, or } x \text{ in second.}\\
x-y=-1 & \text{We will chose } x \text{ in the first}\\
\boldsymbol{x}+y=5 & \text{Solve for the lone variable, subtract } y \text{ from both sides}\\
\underline{\boldsymbol{-y-y}} & \\
x=5-y & \text{Plug into the untouched equation, the second equation}\\
(\boldsymbol{5-y})-y=-1 & \text{Solve, parenthesis are not needed here, combine like terms}\\
5-2y=-1 & \text{Subtract 5 from both sides}\\
\underline{-5 \qquad -5} & \\
\frac{-2y}{-2}=\frac{-6}{-2} & \text{Divide both sides by } -2\\
y=3 & \text{We have our } y!\\
x=5-(3) & \text{Plug into lone variable equation, evaluate}\\
x=2 & \text{Now we have our } x
\end{array}$$

$$(2,3) \quad \text{Our Solution}$$

Just as with graphing it is possible to have no solution $\varnothing$ (parallel lines) or infinite solutions (same line) with the substitution method. While we won't have a parallel line or the same line to look at and conclude if it is one or the other, the process takes an interesting turn as shown in the following example.

Example 174.

$$\begin{array}{rl}
\begin{array}{l}\boldsymbol{y}+4=3x\\ 2y-6x=-8\end{array} & \text{Find the lone variable, } y \text{ in the first equation}\\
y+4=3x & \text{Solve for the lone variable, subtract 4 from both sides}\\
\underline{-4\ \ -4} & \\
y=3x-4 & \text{Plug into untouched equation}\\
2(\boldsymbol{3x-4})-6x=-8 & \text{Solve, distribute through parenthesis}\\
6x-8-6x=-8 & \text{Combine like terms } 6x-6x\\
-8=-8 & \text{Variables are gone! } A \text{ true statement.}\\
\text{Infinite solutions} & \text{Our Solution}
\end{array}$$

Because we had a true statement, and no variables, we know that anything that works in the first equation, will also work in the second equation. However, we do not always end up with a true statement.

Example 175.

$$\begin{array}{rl}
\begin{array}{l}6x-3y=-9\\ -2x+\boldsymbol{y}=5\end{array} & \text{Find the lone variable, } y \text{ in the second equation}\\
-2x+y=5 & \text{Solve for the lone variable, add } 2x \text{ to both sides}\\
\underline{+2x\quad +2x} & \\
y=5+2x & \text{Plug into untouched equation}\\
6x-3(\boldsymbol{5+2x})=-9 & \text{Solve, distribute through parenthesis}\\
6x-15-6x=-9 & \text{Combine like terms } 6x-6x\\
-15\neq-9 & \text{Variables are gone! } A \text{ false statement.}\\
\text{No Solution } \varnothing & \text{Our Solution}
\end{array}$$

Because we had a false statement, and no variables, we know that nothing will work in both equations.

World View Note: French mathematician Rene Descartes wrote a book which included an appendix on geometry. It was in this book that he suggested using letters from the end of the alphabet for unknown values. This is why often we are solving for the variables x, y, and z.

One more question needs to be considered, what if there is no lone variable? If there is no lone variable substitution can still work to solve, we will just have to select one variable to solve for and use fractions as we solve.

Example 176.

$5\boldsymbol{x} - 6y = -14$	No lone variable,
$-2x + 4y = 12$	we will solve for x in the first equation
$5x - 6y = -14$	Solve for our variable, add $6y$ to both sides
$\underline{\mathbf{+6y \quad +6y}}$	
$5x = -14 + 6y$	Divide each term by 5
$\overline{\mathbf{5}} \quad \overline{\mathbf{5}} \quad \overline{\mathbf{5}}$	
$x = \frac{-14}{5} + \frac{6y}{5}$	Plug into untouched equation
$-2\left(\frac{-14}{5} + \frac{6y}{5}\right) + 4y = 12$	Solve, distribute through parenthesis
$\frac{28}{5} - \frac{12y}{5} + 4y = 12$	Clear fractions by multiplying by 5
$\frac{28(\mathbf{5})}{5} - \frac{12y(\mathbf{5})}{5} + 4y(\mathbf{5}) = 12(\mathbf{5})$	Reduce fractions and multiply
$28 - 12y + 20y = 60$	Combine like terms $-12y + 20y$
$28 + 8y = 60$	Subtract 28 from both sides
$\underline{\mathbf{-28 \qquad -28}}$	
$8y = 32$	Divide both sides by 8
$\overline{\mathbf{8}} \quad \overline{\mathbf{8}}$	
$y = 4$	We have our y
$x = \frac{-14}{5} + \frac{6(4)}{5}$	Plug into lone variable equation, multiply
$x = \frac{-14}{5} + \frac{24}{5}$	Add fractions
$x = \frac{10}{5}$	Reduce fraction
$x = 2$	Now we have our x
$(2, 4)$	Our Solution

Using the fractions does make the problem a bit more tricky. This is why we have another method for solving systems of equations that will be discussed in another lesson.

4.2 Practice - Substitution

Solve each system by substitution.

1) $y = -3x$
$y = 6x - 9$

2) $y = x + 5$
$y = -2x - 4$

3) $y = -2x - 9$
$y = 2x - 1$

4) $y = -6x + 3$
$y = 6x + 3$

5) $y = 6x + 4$
$y = -3x - 5$

6) $y = 3x + 13$
$y = -2x - 22$

7) $y = 3x + 2$
$y = -3x + 8$

8) $y = -2x - 9$
$y = -5x - 21$

9) $y = 2x - 3$
$y = -2x + 9$

10) $y = 7x - 24$
$y = -3x + 16$

11) $y = 6x - 6$
$-3x - 3y = -24$

12) $-x + 3y = 12$
$y = 6x + 21$

13) $y = -6$
$3x - 6y = 30$

14) $6x - 4y = -8$
$y = -6x + 2$

15) $y = -5$
$3x + 4y = -17$

16) $7x + 2y = -7$
$y = 5x + 5$

17) $-2x + 2y = 18$
$y = 7x + 15$

18) $y = x + 4$
$3x - 4y = -19$

19) $y = -8x + 19$
$-x + 6y = 16$

20) $y = -2x + 8$
$-7x - 6y = -8$

21) $7x - 2y = -7$
$y = 7$

22) $x - 2y = -13$
$4x + 2y = 18$

23) $x - 5y = 7$
$2x + 7y = -20$

24) $3x - 4y = 15$
$7x + y = 4$

25) $-2x - y = -5$
$x - 8y = -23$

26) $6x + 4y = 16$
$-2x + y = -3$

27) $-6x + y = 20$
$-3x - 3y = -18$

28) $7x + 5y = -13$
$x - 4y = -16$

29) $3x + y = 9$
$2x + 8y = -16$

30) $-5x - 5y = -20$
$-2x + y = 7$

31) $2x + y = 2$
$3x + 7y = 14$

32) $2x + y = -7$
$5x + 3y = -21$

33) $x + 5y = 15$
$-3x + 2y = 6$

34) $2x + 3y = -10$
$7x + y = 3$

35) $-2x + 4y = -16$
$y = -2$

36) $-2x + 2y = -22$
$-5x - 7y = -19$

37) $-6x + 6y = -12$
$8x - 3y = 16$

38) $-8x + 2y = -6$
$-2x + 3y = 11$

39) $2x + 3y = 16$
$-7x - y = 20$

40) $-x - 4y = -14$
$-6x + 8y = 12$

4.3

Systems of Equations - Addition/Elimination

Objective: Solve systems of equations using the addition/elimination method.

When solving systems we have found that graphing is very limited when solving equations. We then considered a second method known as substituion. This is probably the most used idea in solving systems in various areas of algebra. However, substitution can get ugly if we don't have a lone variable. This leads us to our second method for solving systems of equations. This method is known as either Elimination or Addition. We will set up the process in the following examples, then define the five step process we can use to solve by elimination.

Example 177.

$$\begin{array}{rl}
\begin{array}{r} 3x-4y=8 \\ \underline{5x+4y=-24} \end{array} & \text{Notice opposites in front of } y's. \text{ Add columns.} \\
8x \qquad = -16 & \text{Solve for } x, \text{divide by } 8 \\
\overline{8} \qquad\quad \overline{8} & \\
x=-2 & \text{We have our } x! \\
5(-2)+4y=-24 & \text{Plug into either original equation, simplify} \\
-10+4y=-24 & \text{Add 10 to both sides} \\
\underline{+10 \qquad\quad +10} & \\
4y=-14 & \text{Divide by 4} \\
\overline{4} \quad\; \overline{4} & \\
y=\dfrac{-7}{2} & \text{Now we have our } y! \\
\left(-2,\dfrac{-7}{2}\right) & \text{Our Solution}
\end{array}$$

In the previous example one variable had opposites in front of it, $-4y$ and $4y$. Adding these together eliminated the y completely. This allowed us to solve for the x. This is the idea behind the addition method. However, generally we won't have opposites in front of one of the variables. In this case we will manipulate the equations to get the opposites we want by multiplying one or both equations (on both sides!). This is shown in the next example.

Example 178.

$$\begin{array}{rl}
-6x+5y=22 & \text{We can get opposites in front of } x, \text{by multiplying the} \\
2x+3y=2 & \text{second equation by 3, to get } -6x \text{ and } +6x
\end{array}$$

$$\begin{array}{rl}
3(2x+3y)=(2)3 & \text{Distribute to get new second equation.}
\end{array}$$

$$
\begin{array}{rl}
6x+9y=6 & \text{New second equation} \\
\underline{-6x+5y=22} & \text{First equation still the same, add} \\
14y=28 & \text{Divide both sides by 14} \\
\overline{14}\quad\overline{14} & \\
y=2 & \text{We have our } y! \\
2x+3(2)=2 & \text{Plug into one of the original equations, simplify} \\
2x+6=2 & \text{Subtract 6 from both sides} \\
\underline{-6\,-6} & \\
2x=-4 & \text{Divide both sides by 2} \\
\overline{2}\quad\overline{2} & \\
x=-2 & \text{We also have our } x! \\
(-2,2) & \text{Our Solution}
\end{array}
$$

When we looked at the x terms, $-6x$ and $2x$ we decided to multiply the $2x$ by 3 to get the opposites we were looking for. What we are looking for with our opposites is the least common multiple (LCM) of the coefficients. We also could have solved the above problem by looking at the terms with y, $5y$ and $3y$. The LCM of 3 and 5 is 15. So we would want to multiply both equations, the $5y$ by 3, and the $3y$ by -5 to get opposites, $15y$ and $-15y$. This illustrates an important point, some problems we will have to multiply both equations by a constant (on both sides) to get the opposites we want.

Example 179.

$$
\begin{array}{rl}
3x+6y=-9 & \text{We can get opposites in front of } x\text{, find LCM of 6 and 9,} \\
2x+9y=-26 & \text{The LCM is 18. We will multiply to get } 18y \text{ and } -18y
\end{array}
$$

$$
\begin{array}{rl}
3(3x+6y)=(-9)3 & \text{Multiply the first equation by 3, both sides!} \\
9x+18y=-27 &
\end{array}
$$

$$
\begin{array}{rl}
-2(2x+9y)=(-26)(-2) & \text{Multiply the second equation by } -2\text{, both sides!} \\
-4x-18y=52 &
\end{array}
$$

$$
\begin{array}{rl}
9x+18y=-27 & \text{Add two new equations together} \\
\underline{-4x-18y=52} & \\
5x\qquad=25 & \text{Divide both sides by 5} \\
\overline{5}\qquad\quad\overline{5} & \\
x=5 & \text{We have our solution for } x \\
3(5)+6y=-9 & \text{Plug into either original equation, simplify} \\
15+6y=-9 & \text{Subtract 15 from both sides} \\
\underline{-15\qquad\;-15} &
\end{array}
$$

$$\frac{6y}{6} = \frac{-24}{6} \quad \text{Divide both sides by } 6$$
$$y = -4 \quad \text{Now we have our solution for } y$$
$$(5, -4) \quad \text{Our Solution}$$

It is important for each problem as we get started that all variables and constants are lined up before we start multiplying and adding equations. This is illustrated in the next example which includes the five steps we will go through to solve a problem using elimination.

Problem	$2x - 5y = -13$ $-3y + 4 = -5x$
1. Line up the variables and constants	Second Equation: $-3y + 4 = -5x$ $+5x - 4 \quad +5x - 4$ $5x - 3y = -4$
2. Multiply to get opposites (use LCD)	$2x - 5y = -13$ $5x - 3y = -4$ First Equation: multiply by -5 $-5(2x - 5y) = (-13)(-5)$ $-10x + 25y = 65$ Second Equation: multiply by 2 $2(5x - 3y) = (-4)2$ $10x - 6y = -8$ $-10x + 25y = 65$ $10x - 6y = -8$
3. Add	$19y = 57$
4. Solve	$\frac{19y}{19} = \frac{57}{19}$ $y = 3$
5. Plug into either original and solve	$2x - 5(3) = -13$ $2x - 15 = -13$ $+15 \quad +15$ $\frac{2x}{2} = \frac{2}{2}$ $x = 1$
Solution	$(1, 3)$

World View Note: The famous mathematical text, *The Nine Chapters on the Mathematical Art*, which was printed around 179 AD in China describes a formula very similar to Gaussian elimination which is very similar to the addition method.

Just as with graphing and substution, it is possible to have no solution or infinite solutions with elimination. Just as with substitution, if the variables all disappear from our problem, a true statment will indicate infinite solutions and a false statment will indicate no solution.

Example 180.

$$\begin{array}{rl} 2x-5y=3 & \\ -6x+15y=-9 & \text{To get opposites in front of } x\text{, multiply first equation by } 3 \end{array}$$

$$\begin{array}{rl} 3(2x-5y)=(3)3 & \text{Distribute} \\ 6x-15y=9 & \end{array}$$

$$\begin{array}{rl} 6x-15y=9 & \text{Add equations together} \\ \underline{-6x+15y=-9} & \\ 0=0 & \text{True statement} \\ \text{Infinite solutions} & \text{Our Solution} \end{array}$$

Example 181.

$$\begin{array}{rl} 4x-6y=8 & \\ 6x-9y=15 & \text{LCM for } x's \text{ is } 12. \end{array}$$

$$\begin{array}{rl} 3(4x-6y)=(8)3 & \text{Multiply first equation by } 3 \\ 12x-18y=24 & \end{array}$$

$$\begin{array}{rl} -2(6x-9y)=(15)(-2) & \text{Multiply second equation by } -2 \\ -12x+18y=-30 & \end{array}$$

$$\begin{array}{rl} 12x-18y=24 & \text{Add both new equations together} \\ \underline{-12x+18y=-30} & \\ 0=-6 & \text{False statement} \\ \text{No Solution} & \text{Our Solution} \end{array}$$

We have covered three different methods that can be used to solve a system of two equations with two variables. While all three can be used to solve any system, graphing works great for small integer solutions. Substitution works great when we have a lone variable, and addition works great when the other two methods fail. As each method has its own strengths, it is important you are familiar with all three methods.

4.3 Practice - Addition/Elimination

Solve each system by elimination.

1) $4x+2y=0$
$-4x-9y=-28$

2) $-7x+y=-10$
$-9x-y=-22$

3) $-9x+5y=-22$
$9x-5y=13$

4) $-x-2y=-7$
$x+2y=7$

5) $-6x+9y=3$
$6x-9y=-9$

6) $5x-5y=-15$
$5x-5y=-15$

7) $4x-6y=-10$
$4x-6y=-14$

8) $-3x+3y=-12$
$-3x+9y=-24$

9) $-x-5y=28$
$-x+4y=-17$

10) $-10x-5y=0$
$-10x-10y=-30$

11) $2x-y=5$
$5x+2y=-28$

12) $-5x+6y=-17$
$x-2y=5$

13) $10x+6y=24$
$-6x+y=4$

14) $x+3y=-1$
$10x+6y=-10$

15) $2x+4y=24$
$4x-12y=8$

16) $-6x+4y=12$
$12x+6y=18$

17) $-7x+4y=-4$
$10x-8y=-8$

18) $-6x+4y=4$
$-3x-y=26$

19) $5x+10y=20$
$-6x-5y=-3$

20) $-9x-5y=-19$
$3x-7y=-11$

21) $-7x-3y=12$
$-6x-5y=20$

22) $-5x+4y=4$
$-7x-10y=-10$

23) $9x-2y=-18$
$5x-7y=-10$

24) $3x+7y=-8$
$4x+6y=-4$

25) $9x+6y=-21$
$-10x-9y=28$

26) $-4x-5y=12$
$-10x+6y=30$

27) $-7x+5y=-8$
$-3x-3y=12$

28) $8x+7y=-24$
$6x+3y=-18$

29) $-8x-8y=-8$
$10x+9y=1$

30) $-7x+10y=13$
$4x+9y=22$

31) $9y=7-x$
$-18y+4x=-26$

32) $0=-9x-21+12y$
$1+\frac{4}{3}y+\frac{7}{3}x=0$

33) $0=9x+5y$
$y=\frac{2}{7}x$

34) $-6-42y=-12x$
$x-\frac{1}{2}-\frac{7}{2}y=0$

4.4

Systems of Equations - Three Variables

Objective: Solve systems of equations with three variables using addition/elimination.

Solving systems of equations with 3 variables is very similar to how we solve systems with two variables. When we had two variables we reduced the system down to one with only one variable (by substitution or addition). With three variables we will reduce the system down to one with two variables (usually by addition), which we can then solve by either addition or substitution.

To reduce from three variables down to two it is very important to keep the work organized. We will use addition with two equations to eliminate one variable. This new equation we will call (A). Then we will use a different pair of equations and use addition to eliminate the **same** variable. This second new equation we will call (B). Once we have done this we will have two equations (A) and (B) with the same two variables that we can solve using either method. This is shown in the following examples.

Example 182.

$$\begin{aligned} 3x + 2y - z &= -1 \\ -2x - 2y + 3z &= 5 \qquad \text{We will eliminate } y \text{ using two different pairs of equations} \\ 5x + 2y - z &= 3 \end{aligned}$$

$$\begin{array}{rrl}
& 3x + 2y - z = -1 & \text{Using the first two equations,} \\
& \underline{-2x - 2y + 3z = 5} & \text{Add the first two equations} \\
(A) & x \qquad + 2z = 4 & \text{This is equation } (A)\text{, our first equation} \\
\\
& -2x - 2y + 3z = 5 & \text{Using the second two equations} \\
& \underline{5x + 2y - z = 3} & \text{Add the second two equations} \\
(B) & 3x \qquad + 2z = 8 & \text{This is equation } (B)\text{, our second equation} \\
\\
& (A)\ \ x + 2z = 4 & \text{Using } (A) \text{ and } (B) \text{ we will solve this system.} \\
& (B)\ \ 3x + 2z = 8 & \text{We will solve by addition} \\
\\
& -1(x + 2z) = (4)(-1) & \text{Multiply } (A) \text{ by } -1 \\
& -x - 2z = -4 & \\
\\
& -x - 2z = -4 & \text{Add to the second equation, unchanged} \\
& \underline{3x + 2z = 8} & \\
& 2x = 4 & \text{Solve, divide by 2} \\
& \overline{2} \quad \overline{2} & \\
& x = 2 & \text{We now have } x\text{! Plug this into either } (A) \text{ or } (B) \\
\\
& (2) + 2z = 4 & \text{We plug it into } (A)\text{, solve this equation, subtract 2} \\
& \underline{-2 \qquad\quad -2} & \\
& 2z = 2 & \text{Divide by 2} \\
& \overline{2} \quad \overline{2} & \\
& z = 1 & \text{We now have } z\text{! Plug this and } x \text{ into any original equation} \\
\\
& 3(2) + 2y - (1) = -1 & \text{We use the first, multiply } 3(2) = 6 \text{ and combine with } -1 \\
& 2y + 5 = -1 & \text{Solve, subtract 5} \\
& \underline{-5 \quad -5} & \\
& 2y = -6 & \text{Divide by 2} \\
& \overline{2} \quad \overline{2} & \\
& y = -3 & \text{We now have } y\text{!} \\
\\
& (2, -3, 1) & \text{Our Solution}
\end{array}$$

As we are solving for x, y, and z we will have an ordered triplet (x, y, z) instead of just the ordered pair (x, y). In this above problem, y was easily eliminated using the addition method. However, sometimes we may have to do a bit of work to get a variable to eliminate. Just as with addition of two equations, we may have to multiply equations by something on both sides to get the opposites we want so a variable eliminates. As we do this remmeber it is improtant to eliminate the **same** variable both times using two **different** pairs of equations.

Example 183.

$4x-3y+2z=-29$	No variable will easily eliminate.
$6x+2y-z=-16$	We could choose any variable, so we chose x
$-8x-y+3z=23$	We will eliminate x twice.
$4x-3y+2z=-29$	Start with first two equations. LCM of 4 and 6 is 12.
$6x+2y-z=-16$	Make the first equation have $12x$, the second $-12x$
$3(4x-3y+2z)=(-29)3$	Multiply the first equation by 3
$12x-9y+6z=-87$	
$-2(6x+2y-z)=(-16)(-2)$	Multiply the second equation by -2
$-12x-4y+2z=32$	
$12x-9y+6z=-87$	Add these two equations together
$\underline{-12x-4y+2z=32}$	
$(A)\quad -13y+8z=-55$	This is our (A) equation
$6x+2y-z=-16$	Now use the second two equations (a different pair)
$-8x-y+3z=23$	The LCM of 6 and -8 is 24.
$4(6x+2y-z)=(-16)4$	Multiply the first equation by 4
$24x+8y-4=-64$	
$3(-8x-y+3z)=(23)3$	Multiply the second equation by 3
$-24x-3y+9z=69$	
$24x+8y-4=-64$	Add these two equations together
$\underline{-24x-3y+9z=69}$	
$(B)\quad 5y+5z=5$	This is our (B) equation

$$\begin{array}{rll} (A) & -13y+8z=-55 & \text{Using } (A) \text{ and } (B) \text{ we will solve this system} \\ (B) & 5y+5z=5 & \text{The second equation is solved for } z \text{ to use substitution} \end{array}$$

$$\begin{array}{rl} 5y+5z=5 & \text{Solving for } z\text{, subtract } 5y \\ \underline{-5y \qquad -5y} & \\ 5z=5-5y & \text{Divide each term by } 5 \\ \overline{5} \quad \overline{5} \quad \overline{5} & \\ z=1-y & \text{Plug into untouched equation} \\ -13y+8(1-y)=-55 & \text{Distribute} \\ -13y+8-8y=-55 & \text{Combine like terms } -13y-8y \\ -21y+8=-55 & \text{Subtract } 8 \\ \underline{-8 \quad -8} & \\ -21y=-63 & \text{Divide by } -21 \\ \overline{-21} \quad \overline{-21} & \\ y=3 & \text{We have our } y\text{! Plug this into } z= \text{ equations} \\ z=1-(3) & \text{Evaluate} \\ z=-2 & \text{We have } z\text{, now find } x \text{ from original equation.} \end{array}$$

$$\begin{array}{rl} 4x-3(3)+2(-2)=-29 & \text{Multiply and combine like terms} \\ 4x-13=-29 & \text{Add } 13 \\ \underline{+13 \quad +13} & \\ 4x=-16 & \text{Divide by } 4 \\ \overline{4} \quad \overline{4} & \\ x=-4 & \text{We have our } x\text{!} \end{array}$$

$$\begin{array}{rl} (-4,3,-2) & \text{Our Solution!} \end{array}$$

World View Note: Around 250, *The Nine Chapters on the Mathematical Art* were published in China. This book had 246 problems, and chapter 8 was about solving systems of equations. One problem had four equations with five variables!

Just as with two variables and two equations, we can have special cases come up with three variables and three equations. The way we interpret the result is identical.

Example 184.

$$\begin{array}{rl} 5x-4y+3z=-4 & \\ -10x+8y-6z=8 & \text{We will eliminate } x\text{, start with first two equations} \end{array}$$

$$15x - 12y + 9z = -12$$

$$\begin{aligned} 5x - 4y + 3z &= -4 && \text{LCM of 5 and } -10 \text{ is } 10. \\ -10x + 8y - 6z &= 8 \end{aligned}$$

$$\begin{aligned} 2(5x - 4y + 3z) &= -4(2) && \text{Multiply the first equation by 2} \\ 10x - 8y + 6z &= -8 \end{aligned}$$

$$\begin{aligned} 10x - 8y + 6z &= -8 && \text{Add this to the second equation, unchanged} \\ \underline{-10x + 8y - 6} &\underline{= 8} \\ 0 &= 0 && A \text{ true statment} \\ \text{Infinite Solutions} & && \text{Our Solution} \end{aligned}$$

Example 185.

$$\begin{aligned} 3x - 4y + z &= 2 \\ -9x + 12y - 3z &= -5 && \text{We will eliminate } z\text{, starting with the first two equations} \\ 4x - 2y - z &= 3 \end{aligned}$$

$$\begin{aligned} 3x - 4y + z &= 2 && \text{The LCM of 1 and } -3 \text{ is } 3 \\ -9x + 12y - 3z &= -5 \end{aligned}$$

$$\begin{aligned} 3(3x - 4y + z) &= (2)3 && \text{Multiply the first equation by 3} \\ 9x - 12y + 3z &= 6 \end{aligned}$$

$$\begin{aligned} 9x - 12y + 3z &= 6 && \text{Add this to the second equation, unchanged} \\ \underline{-9x + 12y - 3z} &\underline{= -5} \\ 0 &= 1 && A \text{ false statement} \\ \text{No Solution } \varnothing & && \text{Our Solution} \end{aligned}$$

Equations with three (or more) variables are not any more difficult than two variables if we are careful to keep our information organized and eliminate the same variable twice using two different pairs of equations. It is possible to solve each system several different ways. We can use different pairs of equations or eliminate variables in different orders, but as long as our information is organized and our algebra is correct, we will arrive at the same final solution.

4.4 Practice - Three Variables

Solve each of the following systems of equation.

1) $a-2b+c=5$
$2a+b-c=-1$
$3a+3b-2c=-4$

2) $2x+3y=z-1$
$3x=8z-1$
$5y+7z=-1$

3) $3x+y-z=11$
$x+3y=z+13$
$x+y-3z=11$

4) $x+y+z=2$
$6x-4y+5z=31$
$5x+2y+2z=13$

5) $x+6y+3z=4$
$2x+y+2z=3$
$3x-2y+z=0$

6) $x-y+2z=-3$
$x+2y+3z=4$
$2x+y+z=-3$

7) $x+y+z=6$
$2x-y-z=-3$
$x-2y+3z=6$

8) $x+y-z=0$
$x+2y-4z=0$
$2x+y+z=0$

9) $x+y-z=0$
$x-y-z=0$
$x+y+2z=0$

10) $x+2y-z=4$
$4x-3y+z=8$
$5x-y=12$

11) $-2x+y-3z=1$
$x-4y+z=6$
$4x+16y+4z=24$

12) $4x+12y+16z=4$
$3x+4y+5z=3$
$x+8y+11z=1$

13) $2x+y-3z=0$
$x-4y+z=0$
$4x+16y+4z=0$

14) $4x+12y+16z=0$
$3x+4y+5z=0$
$x+8y+11z=0$

15) $3x+2y+2z=3$
$x+2y-z=5$
$2x-4y+z=0$

16) $p+q+r=1$
$p+2q+3r=4$
$4p+5q+6r=7$

17) $x-2y+3z=4$
$2x-y+z=-1$
$4x+y+z=1$

18) $x+2y-3z=9$
$2x-y+2z=-8$
$3x-y-4z=3$

19) $x-y+2z=0$
$x-2y+3z=-1$
$2x-2y+z=-3$

20) $4x-7y+3z=1$
$3x+y-2z=4$
$4x-7y+3z=6$

21) $4x-3y+2z=40$
$5x+9y-7z=47$
$9x+8y-3z=97$

22) $3x+y-z=10$
$8x-y-6z=-3$
$5x-2y-5z=1$

23) $3x + 3y - 2z = 13$
$6x + 2y - 5z = 13$
$5x - 2y - 5z = -1$

24) $2x - 3y + 5z = 1$
$3x + 2y - z = 4$
$4x + 7y - 7z = 7$

25) $3x - 4y + 2z = 1$
$2x + 3y - 3z = -1$
$x + 10y - 8z = 7$

26) $2x + y = z$
$4x + z = 4y$
$y = x + 1$

27) $m + 6n + 3p = 8$
$3m + 4n = -3$
$5m + 7n = 1$

28) $3x + 2y = z + 2$
$y = 1 - 2x$
$3z = -2y$

29) $-2w + 2x + 2y - 2z = -10$
$w + x + y + z = -5$
$3w + 2x + 2y + 4z = -11$
$w + 3x - 2y + 2z = -6$

30) $-w + 2x - 3y + z = -8$
$-w + x + y - z = -4$
$w + x + y + z = 22$
$-w + x - y - z = -14$

31) $w + x + y + z = 2$
$w + 2x + 2y + 4z = 1$
$-w + x - y - z = -6$
$-w + 3x + y - z = -2$

32) $w + x - y + z = 0$
$-w + 2x + 2y + z = 5$
$-w + 3x + y - z = -4$
$-2w + x + y - 3z = -7$

4.5

Solving Systems with Matrices

Objectives:

1. **Solve a system of linear equations using matrices.**

Writing the Augmented Matrix of a System of Equations

A **matrix** is an array of numbers can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an **augmented matrix**.

For example, consider the following 2 × 2 system of equations.

$$\begin{aligned} 3x + 4y &= 7 \\ 4x - 2y &= 5 \end{aligned}$$

We can write this system as an augmented matrix:

$$\left[\begin{array}{rr|r} 3 & 4 & 7 \\ 4 & -2 & 5 \end{array}\right]$$

A three-by-three **system of equations** such as

$$\begin{aligned} 3x - y - z &= 0 \\ x + y &= 5 \\ 2x - 3z &= 2 \end{aligned}$$

can be represented by the augmented matrix

$$\left[\begin{array}{rrr|r} 3 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2 \end{array}\right]$$

Notice that the matrix is written so that the variables line up in their own columns: x-terms go in the first column, y-terms in the second column, and z-terms in the third column. It is very important that each equation is written in standard form $ax + by + cz = d$ so that the variables line up. When there is a missing variable term in an equation, the coefficient is 0.

The calculator can be used to solve systems of linear equations by performing operations on the rows of augmented matrix to change the matrix to **row reduced echelon form**. The following matrices are in row reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Notice that the coefficients are all zero with a diagonal of ones. This form of the matrix makes it easy to see the solution to the system. Translating each row of the matrix into the equation that it represents shows us that the solution to the system is easy to identify.

Row Reduced Echelon Form		**Equivalent Equations**		**Simplified Equations**		**Solution to the System**
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \end{bmatrix}$	$\rightarrow$	$\begin{matrix} 1x + 0y = 0 \\ 0x + 1y = 6 \end{matrix}$	$\rightarrow$	$\begin{matrix} x + 0 = 0 \\ 0 + y = 6 \end{matrix}$	$\rightarrow$	$\begin{matrix} x = 0 \\ y = 6 \end{matrix}$
$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$	$\rightarrow$	$\begin{matrix} 1x + 0y + 0z = 3 \\ 0x + 1y + 0z = -2 \\ 0x + 0y + 1z = 7 \end{matrix}$	$\rightarrow$	$\begin{matrix} x + 0 + 0 = 3 \\ 0 + y + 0 = -2 \\ 0 + 0 + z = 7 \end{matrix}$	$\rightarrow$	$\begin{matrix} x = 3 \\ y = -2 \\ z = 7 \end{matrix}$

Notice how the last column in the row reduced echelon form corresponds to the solution to the system.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \end{bmatrix} \rightarrow \begin{matrix} x = 0 \\ y = 6 \end{matrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{bmatrix} \rightarrow \begin{matrix} x = 3 \\ y = -2 \\ z = 7 \end{matrix}$$

The operations for changing an augmented matrix to its row reduced echelon form are outside the scope of this class. However, the calculator is capable of performing these operations.

There are four steps to reducing a matrix to row reduced echelon form.

1. Change both equation to standard form $Ax + By = C$,
2. Represent the system with an augmented matrix,
3. Enter the matrix into the calculator,
4. Convert the matrix to row reduced echelon form.

Example 1: Solve the following system of equations.

$$\begin{aligned} y &= -3x + 14 \\ 2x - y &= 6 \end{aligned}$$

Answer:

Step 1: Change both equations to standard form.

First, represent the system with an augmented matrix. To do this, change each equation in the system to standard form Ax + By = C. The first equation $y = -3x + 14$ is not in standard form, so add 3x to both sides of the equation.

$$\begin{array}{rcl} & y = & -3x + 14 \\ +3x & & +3x \\ 3x & + y = & 14 \end{array}$$

The second equation $2x - y = 6$ is already in standard form. The two equations in standard form are:

$$\begin{aligned} 3x + y &= 14 \\ 2x - y &= 6 \end{aligned}$$

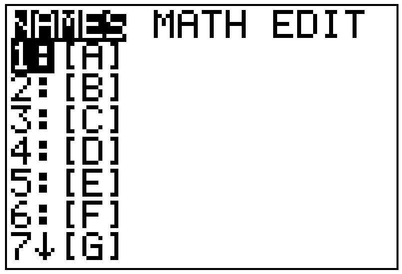
Step 2: Represent the system with an augmented matrix.

Now represent this system with an augmented matrix. Remember to place all the x coefficients in the first column and the y coefficients in the second column.

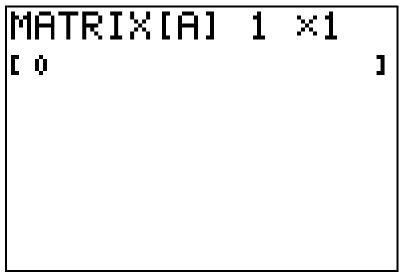
$$\begin{aligned} 3x + y &= 14 \\ 2x - y &= 6 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} 3 & 1 & 14 \\ 2 & -1 & 6 \end{bmatrix}$$

Step 3: Enter the matrix into the calculator.

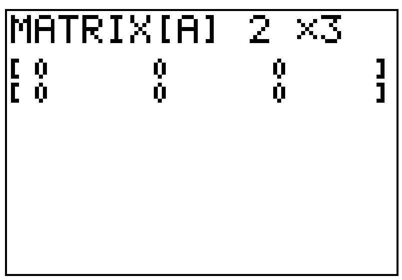
To enter the matrix into the calculator, either press the **MATRX** button, or if the calculator does not have a **MATRX** button press the **2ND key** and then the $\boldsymbol{x^{-1}}$ key to enter the **MATRIX** menu. The calculator screen should look similar to the image below.



Use the **right arrow** key to cursor over to **EDIT** with the **right arrow** key. Press **1** to edit matrix [A]. (Note: the letter A is just a label for a matrix. There is space enough for ten matrices on your calculator, labeled A through J).



Enter the size of the matrix. The matrix for this system has 2 rows and 3 columns, so press **2** and **ENTER**, then **3** and **ENTER**.



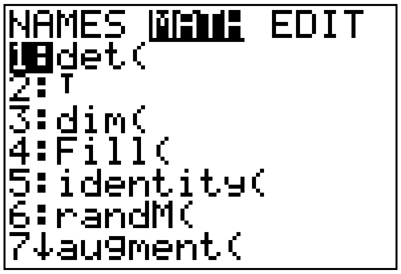
Enter the coefficients and constants in the augmented matrix into the calculator. After entering a number, press **ENTER** to move to the next entry. You can also use the arrow keys to navigate from entry to entry.

$$\begin{aligned} 3x + y &= 14 \\ 2x - y &= 6 \end{aligned} \rightarrow \begin{bmatrix} 3 & 1 & 14 \\ 2 & -1 & 6 \end{bmatrix} \rightarrow$$

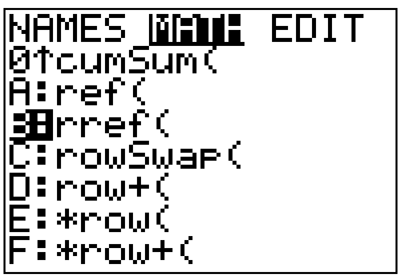
Step 4: Convert the matrix to row reduced echelon form.

Now that we have entered the matrix into the calculator, we want to reduce it to row reduced echelon form. Press **2ND** and **MODE** to **QUIT**.

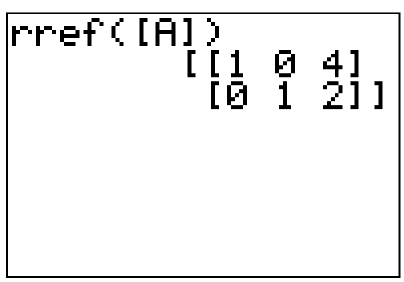
Next press **2ND** and $\boldsymbol{x^{-1}}$ (or press the **MATRX** button) to enter the **MATRIX** menu. Then curser right to the **MATH** menu.



Use the arrow keys to highlight **rref** (short for row reduced echelon form) and press **ENTER**.



The last step is to input the name of the matrix we want to row reduce into the calculator. Press **2ND** and $\boldsymbol{x^{-1}}$ (or press the MATRX button) to enter the **MATRIX** menu. Then press **1** to select matrix A. Close the parentheses and press **ENTER.**



The first row of the matrix represents the equation

$$1x + 0y = 4.$$

Which can be simplified to

$$x + 0 = 4$$

or

$$\boldsymbol{x = 4.}$$

Similarly, the second row of the matrix represents the equation

$$0x + 1y = 2 \quad \text{or} \quad \boldsymbol{y = 2.}$$

The solutions to the system is $x = 4$ and $y = 2$ or **(4, 2).**

Example 2: Solve the following system of equations.

$$4x + 5y = 31$$
$$x = 10y - 26$$

Answer:

Change both equations to standard form A*x* + B*y* = C.

$$4x + 5y = 31$$
$$x - 10y = -26$$

Write the corresponding augmented matrix.

$$\begin{bmatrix} 4 & 5 & 31 \\ 1 & -10 & -26 \end{bmatrix}$$

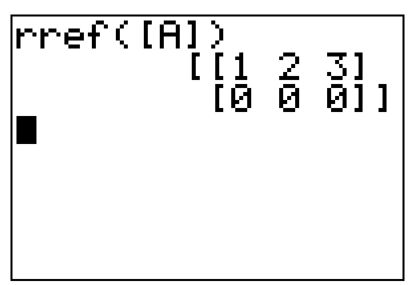
Use the steps outlined in **Example 1** to find the row reduced echelon form of the matrix.

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

The solutions to the system is $x = 4$ and $y = 3$ or **(4, 3).**

Dependent and Inconsistent Systems

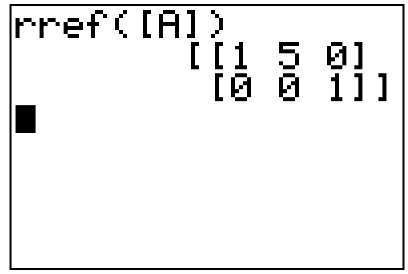
If a system was **dependent**, the calculator would display a row of all 0's, like the image to the right.



The last row translates to $0x + 0y = 0$, which is always true and the solution is that there are **infinitely many solutions.**

If the system was **inconsistent**, the calculator would display a row of 0's with a 1 in the right column, like in this image.

That last row represents the equation $0x + 0y = 1$, which is impossible and the system has **no solution**.



4.4 Practice – Solving Systems with Matrices

For the following exercises, write the augmented matrix for the linear system.

6. $8x - 37y = 8$
$2x + 12y = 3$

7. $16y = 4$
$9x - y = 2$

8. $3x + 2y + 10z = 3$
$-6x + 2y + 5z = 13$
$4x + z = 18$

9. $x + 5y + 8z = 19$
$12x + 3y = 4$
$3x + 4y + 9z = -7$

10. $6x + 12y + 16z = 4$
$19x - 5y + 3z = -9$
$x + 2y = -8$

For the following exercises, write the linear system from the augmented matrix.

11. $\left[\begin{array}{cc|c} -2 & 5 & 5 \\ 6 & -18 & 26 \end{array}\right]$

12. $\left[\begin{array}{cc|c} 3 & 4 & 10 \\ 10 & 17 & 439 \end{array}\right]$

13. $\left[\begin{array}{ccc|c} 3 & 2 & 0 & 3 \\ -1 & -9 & 4 & -1 \\ 8 & 5 & 7 & 8 \end{array}\right]$

14. $\left[\begin{array}{ccc|c} 8 & 29 & 1 & 43 \\ -1 & 7 & 5 & 38 \\ 0 & 0 & 3 & 10 \end{array}\right]$

15. $\left[\begin{array}{ccc|c} 4 & 5 & -2 & 12 \\ 0 & 1 & 58 & 2 \\ 8 & 7 & -3 & -5 \end{array}\right]$

For the following exercises, solve the system by Gaussian elimination.

16. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right]$

17. $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 0 & 2 \end{array}\right]$

18. $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$

19. $\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 4 & -5 & 6 \end{array}\right]$

20. $\left[\begin{array}{cc|c} -2 & 0 & 1 \\ 0 & 2 & -1 \end{array}\right]$

21. $2x - 3y = -9$
$5x + 4y = 58$

22. $6x + 2y = -4$
$3x + 4y = -17$

23. $2x + 3y = 12$
$4x + y = 14$

24. $-4x - 3y = -2$
$3x - 5y = -13$

25. $-5x + 8y = 3$
$10x + 6y = 5$

26. $3x + 4y = 12$
$-6x - 8y = -24$

27. $-60x + 45y = 12$
$20x - 15y = -4$

28. $11x + 10y = 43$
$15x + 20y = 65$

29. $2x - y = 2$
$3x + 2y = 17$

30. $-1.06x - 2.25y = 5.51$
$-5.03x - 1.08y = 5.40$

31. $\frac{3}{4}x - \frac{3}{5}y = 4$
$\frac{1}{4}x + \frac{2}{3}y = 1$

32. $\frac{1}{4}x - \frac{2}{3}y = -1$
$\frac{1}{2}x + \frac{1}{3}y = 3$

33. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 31 \\ 0 & 1 & 1 & 45 \\ 0 & 0 & 1 & 87 \end{array}\right]$

34. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 50 \\ 1 & 1 & 0 & 20 \\ 0 & 1 & 1 & -90 \end{array}\right]$

35. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{array}\right]$

36. $\left[\begin{array}{ccc|c} -0.1 & 0.3 & -0.1 & 0.2 \\ -0.4 & 0.2 & 0.1 & 0.8 \\ 0.6 & 0.1 & 0.7 & -0.8 \end{array}\right]$

37. $\begin{aligned} -2x + 3y - 2z &= 3 \\ 4x + 2y - z &= 9 \\ 4x - 8y + 2z &= -6 \end{aligned}$

38. $\begin{aligned} x + y - 4z &= -4 \\ 5x - 3y - 2z &= 0 \\ 2x + 6y + 7z &= 30 \end{aligned}$

39. $\begin{aligned} 2x + 3y + 2z &= 1 \\ -4x - 6y - 4z &= -2 \\ 10x + 15y + 10z &= 5 \end{aligned}$

40. $\begin{aligned} x + 2y - z &= 1 \\ -x - 2y + 2z &= -2 \\ 3x + 6y - 3z &= 5 \end{aligned}$

41. $\begin{aligned} x + 2y - z &= 1 \\ -x - 2y + 2z &= -2 \\ 3x + 6y - 3z &= 3 \end{aligned}$

42. $\begin{aligned} x + y &= 2 \\ x + z &= 1 \\ -y - z &= -3 \end{aligned}$

43. $\begin{aligned} x + y + z &= 100 \\ x + 2z &= 125 \\ -y + 2z &= 25 \end{aligned}$

44. $\begin{aligned} \frac{1}{4}x - \frac{2}{3}z &= -\frac{1}{2} \\ \frac{1}{5}x + \frac{1}{3}y &= \frac{4}{7} \\ \frac{1}{5}y - \frac{1}{3}z &= \frac{2}{9} \end{aligned}$

45. $\begin{aligned} -\frac{1}{2}x + \frac{1}{2}y + \frac{1}{7}z &= -\frac{53}{14} \\ \frac{1}{2}x - \frac{1}{2}y + \frac{1}{4}z &= 3 \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{3}z &= \frac{23}{15} \end{aligned}$

46. $\begin{aligned} -\frac{1}{2}x - \frac{1}{3}y + \frac{1}{4}z &= -\frac{29}{6} \\ \frac{1}{5}x + \frac{1}{6}y - \frac{1}{7}z &= \frac{431}{210} \\ -\frac{1}{8}x + \frac{1}{9}y + \frac{1}{10}z &= -\frac{49}{45} \end{aligned}$

For the following exercises, use Gaussian elimination to solve the system.

47. $\begin{aligned} \frac{x-1}{7} + \frac{y-2}{8} + \frac{z-3}{4} &= 0 \\ x + y + z &= 6 \\ \frac{x+2}{3} + 2y + \frac{z-3}{3} &= 5 \end{aligned}$

48. $\begin{aligned} \frac{x-1}{4} - \frac{y+1}{4} + 3z &= -1 \\ \frac{x+5}{2} + \frac{y+7}{4} - z &= 4 \\ x + y - \frac{z-2}{2} &= 1 \end{aligned}$

49. $\begin{aligned} \frac{x-3}{4} - \frac{y-1}{3} + 2z &= -1 \\ \frac{x+5}{2} + \frac{y+5}{2} + \frac{z+5}{2} &= 8 \\ x + y + z &= 1 \end{aligned}$

50. $\begin{aligned} \frac{x-3}{10} + \frac{y+3}{2} - 2z &= 3 \\ \frac{x+5}{4} - \frac{y-1}{8} + z &= \frac{3}{2} \\ \frac{x-1}{4} + \frac{y+4}{2} + 3z &= \frac{3}{2} \end{aligned}$

51. $\begin{aligned} \frac{x-3}{4} - \frac{y-1}{3} + 2z &= -1 \\ \frac{x+5}{2} + \frac{y+5}{2} + \frac{z+5}{2} &= 7 \\ x + y + z &= 1 \end{aligned}$

4.6

Applications of Linear Systems

Objectives:

1. Set up and solve applications involving relationships between numbers.
2. Set up and solve applications involving interest and money.
3. Set up and solve mixture problems.
4. Set up and solve uniform motion problems (distance problems).

Problems Involving Relationships between Real Numbers

We now have the techniques needed to solve linear systems. For this reason, we are no longer limited to using one variable when setting up equations that model applications. If we translate an application to a mathematical setup using two variables, then we need to form a linear system with two equations.

Example 1: The sum of two numbers is 40 and their difference is 8. Find the numbers.

Solution:

Identify variables.

Let x represent one of the unknown numbers.
Let y represent the other unknown number.

Set up equations: When using two variables, we need to set up two equations. The first key phrase, "the*sum* of the two numbers is 40," translates as follows:

$$x + y = 40$$

And the second key phrase, "the *difference* is 8," leads us to the second equation:

$$x - y = 8$$

Therefore, our algebraic setup consists of the following system:

$$\begin{cases} x + y = 40 \\ x - y = 8 \end{cases}$$

Solve: We can solve the resulting system using any method of our choosing. Here we choose to solve by elimination. Adding the equations together eliminates the variable *y*.

$$\begin{array}{rl} & x + y = 40 \\ + & x - y = 8 \\ \hline & 2x \quad = 48 \\ & x = 24 \end{array}$$

Once we have *x*, back substitute to find *y*.

$$\begin{aligned} x + y &= 40 \\ 24 + y &= 40 \\ 24 + y - 24 &= 40 - 24 \\ y &= 16 \end{aligned}$$

Check: The sum of the two numbers should be 42 and their difference 8.

$$\begin{aligned} 24 + 16 &= 40 \\ 24 - 16 &= 8 \end{aligned}$$

Answer: The two numbers are 24 and 16.

Example 2: The sum of 9 times a larger number and twice a smaller is 6. The difference of 3 times the larger and the smaller is 7. Find the numbers.

Solution: Begin by assigning variables to the larger and smaller number.

Let x represent the larger number.
Let y represent the smaller number.

The first sentence describes a sum and the second sentence describes a difference.

$$\overbrace{9x}^{\text{9 times a larger}} + \overbrace{2y}^{\text{twice a smaller}} = 6$$

$$\overbrace{3x}^{\text{3 times the larger}} - \overbrace{y}^{\text{the smaller}} = 7$$

This leads to the following system:

$$\begin{cases} 9x + 2y = 6 \\ 3x - y = 7 \end{cases}$$

Solve using the elimination method. Multiply the second equation by 2 and add.

$$\begin{cases} 9x+2y=6 \\ 3x-y=7 \end{cases} \underset{\times 2}{\Rightarrow} \begin{cases} 9x+2y=6 \\ 6x-2y=14 \end{cases}$$

$$\begin{array}{rl} & 9x+2y=6 \\ + & 6x-2y=14 \\ \hline & 15x \quad\; = 20 \\ & x=\dfrac{20}{15} \\ & x=\dfrac{4}{3} \end{array}$$

Back substitute to find y.

$$\begin{aligned} 3x - y &= 7 \\ 3\left(\frac{4}{3}\right) - y &= 7 \\ 4 - y &= 7 \\ -y &= 3 \\ y &= -3 \end{aligned}$$

Answer: The larger number is 4/3 and the smaller number is −3.

Try this! The sum of two numbers is 3. When twice the smaller number is subtracted from 6 times the larger the result is 22. Find the numbers.

Answer: The two numbers are −1/2 and 7/2.

Video Solution (click to see video)

Interest and Money Problems

In this section, the interest and money problems should seem familiar. The difference is that we will be making use of two variables when setting up the algebraic equations.

Example 3: A roll of 32 bills contains only \$5 bills and \$10 bills. If the value of the roll is \$220, then how many of each bill are in the roll?

Solution: Begin by identifying the variables.

Let x represent the number of \$5 bills.
Let y represent the number of \$10 bills.

When using two variables, we need to set up two equations. The first equation is created from the fact that there are 32 bills.

$$x + y = 32$$

The second equation sums the value of each bill: the total value is \$220.

$$\$5 \cdot x + \$10 \cdot y = \$220$$

Present both equations as a system; this is our algebraic setup.

$$\begin{cases} x + y = 32 \\ 5x + 10y = 220 \end{cases}$$

Here we choose to solve by elimination, although substitution would work just as well. Eliminate x by multiplying the first equation by -5.

$$\begin{cases} x + y = 32 \\ 5x + 10y = 220 \end{cases} \overset{\times(-5)}{\Longrightarrow} \begin{cases} -5x - 5y = -160 \\ 5x + 10y = 220 \end{cases}$$

Now add the equations together:

$$\begin{array}{rl} -5x-5y & =-160 \\ +\quad 5x+10y & =220 \\ \hline 5y & =60 \\ \dfrac{5y}{5} & =\dfrac{60}{5} \\ y & =12 \end{array}$$

Once we have *y*, the number of \$10 bills, back substitute to find *x*.

$$\begin{aligned} x+y &= 32 \\ x+12 &= 32 \\ x+12-12 &= 32-12 \\ x &= 20 \end{aligned}$$

Answer: There are twenty \$5 bills and twelve \$10 bills. The check is left to the reader.

Example 4: A total of \$6,300 was invested in two accounts. Part was invested in a CD at a 412%annual interest rate and part was invested in a money market fund at a 334% annual interest rate. If the total simple interest for one year was \$267.75, then how much was invested in each account?

Solution:

Let x represent the amount invested at $4\frac{1}{2}\% = 4.5\% = 0.045$

Let y represent the amount invested at $3\frac{3}{4}\% = 3.75\% = 0.0375$

The total amount in both accounts can be expressed as

$$x + y = 6,300$$

To set up a second equation, use the fact that the total interest was \$267.75. Recall that the interest for one year is the interest rate times the principal (I=prt=pr·1=pr). Use this to add the interest in both accounts. Be sure to use the decimal equivalents for the interest rates given as percentages.

$$\begin{array}{ccccc} \textit{interest from the CD} & + & \textit{interest from the fund} & = & \textit{total interest} \\ 0.045x & + & 0.0375y & = & 267.75 \end{array}$$

These two equations together form the following linear system:

$$\begin{cases} x + y = 6,300 \\ 0.045x + 0.0375y = 267.75 \end{cases}$$

Eliminate y by multiplying the first equation by -0.0375.

$$\begin{cases} x + y = 6,300 \\ 0.045x + 0.0375y = 267.75 \end{cases} \overset{\times(-0.0375)}{\Rightarrow} \begin{cases} -0.0375x - 0.0375y = -236.25 \\ 0.045x + 0.0375y = 267.75 \end{cases}$$

Next, add the equations together to eliminate the variable y.

$$\begin{array}{rl} -0.0375x - 0.0375y &= -236.25 \\ + \quad 0.045x + 0.0375y &= 267.75 \\ \hline 0.0075x \qquad\qquad &= 31.5 \\ \dfrac{0.0075x}{0.0075} &= \dfrac{31.5}{0.0075} \\ x &= 4,200 \end{array}$$

Back substitute.

$$\begin{aligned} x + y &= 6,300 \\ 4,200 + y &= 6,300 \\ 4,200 + y - 4,200 &= 6,300 - 4,200 \\ y &= 2,100 \end{aligned}$$

Answer: $4,200 was invested at 412% and $2,100 was invested at 334%.

At this point, we should be able to solve these types of problems in two ways: with one variable and now with two variables. Setting up word problems with two variables often simplifies the entire process, particularly when the relationships between the variables are not so clear.

Try this! On the first day of a two-day meeting, 10 coffees and 10 doughnuts were purchased for a total of $20.00. Since nobody drank the coffee and all the doughnuts were eaten, the next day only 2 coffees and 14 doughnuts were purchased for a total of $13.00. How much did each coffee and each doughnut cost?

Answer: Coffee: $1.25; doughnut: $0.75

Video Solution (click to see video)

Mixture Problems

Mixture problems often include a percentage and some total amount. It is important to make a distinction between these two types of quantities. For example, if a problem states that a 20-ounce container is filled with a 2% saline (salt) solution, then this means that the container is filled with a mixture of salt and water as follows:

	Percentage	**Amount**
Salt	2% = 0.02	0.02(20 ounces) = 0.4 ounces
Water	98% = 0.98	0.98(20 ounces) = 19.6 ounces

In other words, we multiply the percentage times the total to get the amount of each part of the mixture.

Example 5: A 2% saline solution is to be combined and mixed with a 5% saline solution to produce 72 ounces of a 2.5% saline solution. How much of each is needed?

Solution:

Let x represent the amount of 2% saline solution needed.
Let y represent the amount of 5% saline solution needed.

The total amount of saline solution needed is 72 ounces. This leads to one equation,

$$x + y = 72$$

The second equation adds up the amount of salt in the correct percentages. The amount of salt is obtained by multiplying the percentage times the amount, where the variables *x* and *y* represent the amounts of the solutions.

$$\begin{array}{ccccc} \textit{salt in 2\% solution} & + & \textit{salt in 5\% solution} & = & \textit{salt in the end solution} \\ 0.02x & + & 0.05y & = & 0.025(72) \end{array}$$

The algebraic setup consists of both equations presented as a system:

$$\begin{cases} x + y = 72 \\ 0.02x + 0.05y = 0.025(72) \end{cases}$$

Solve.

$$\begin{cases} x + y = 72 \\ 0.02x + 0.05y = 0.025(72) \end{cases} \overset{\times(-0.02)}{\Rightarrow} \begin{cases} -0.02x - 0.02y = -1.44 \\ 0.02x + 0.05y = 1.8 \end{cases}$$

$$\begin{array}{rr} & -0.02x - 0.02y = -1.44 \\ + & 0.02x + 0.05y = 1.8 \\ \hline & 0.03y = 0.36 \\ & \dfrac{0.03y}{0.03} = \dfrac{0.36}{0.03} \\ & y = 12 \end{array}$$

Back substitute.

$$\begin{aligned} x + y &= 72 \\ x + 12 &= 72 \\ x + 12 - 12 &= 72 - 12 \\ x &= 60 \end{aligned}$$

Answer: We need 60 ounces of the 2% saline solution and 12 ounces of the 5% saline solution.

Example 6: A 50% alcohol solution is to be mixed with a 10% alcohol solution to create an 8-ounce mixture of a 32% alcohol solution. How much of each is needed?

Solution:

Let x represent the amount of 50% alcohol solution needed.
Let y represent the amount of 10% alcohol solution needed.

The total amount of the mixture must be 8 ounces.

$$x + y = 8$$

The second equation adds up the amount of alcohol from each solution in the correct percentages. The amount of alcohol in the end result is 32% of 8 ounces, or 0.032(8).

$$\begin{array}{ccccc} \textit{alcohol in } 50\% \textit{ solution} & + & \textit{alcohol in } 10\% \textit{ solution} & = & \textit{alcohol in the end solution} \\ 0.50x & + & 0.10y & = & 0.32(8) \end{array}$$

Now we can form a system of two linear equations and two variables as follows:

$$\begin{cases} x + y = 8 \\ 0.50x + 0.10y = 0.32(8) \end{cases}$$

In this example, multiply the second equation by 100 to eliminate the decimals. In addition, multiply the first equation by −10 to line up the variable y to eliminate.

$$Equation\ 1:$$

$$-10\,(x+y)=-10\,(8)$$

$$-10x-10y=-80$$

$$Equation\ 2:$$

$$100\ \ 0.50x+0.10y\ \ =100\,(0.32)(8)$$

$$50x+10y=256$$

We obtain the following equivalent system:

$$\begin{cases} x+y=8 \\ 0.50x+0.10y=0.32(8) \end{cases} \begin{array}{c} \overset{\times(-10)}{\Rightarrow} \\ \underset{\times 100}{\Rightarrow} \end{array} \begin{cases} -10x-10y=-80 \\ 50x+10y=256 \end{cases}$$

Add the equations and then solve for *x*:

$$\begin{array}{rrr} & -10x-10y & =-80 \\ + & 50x+10y & =256 \\ \hline & 40x & =176 \end{array}$$

$$\frac{40x}{40}=\frac{176}{40}$$

$$x=4.4$$

Back substitute.

$$x+y=8$$

$$4.4+y=8$$

$$4.4+y-4.4=8-4.4$$

$$x=3.6$$

Answer: To obtain 8 ounces of a 32% alcohol mixture we need to mix 4.4 ounces of the 50% alcohol solution and 3.6 ounces of the 10% solution.

Try this! A 70% antifreeze concentrate is to be mixed with water to produce a 5-gallon mixture containing 28% antifreeze. How much water and antifreeze concentrate is needed?

Answer: We need to mix 3 gallons of water with 2 gallons of antifreeze concentrate.

Video Solution (click to see video)

Uniform Motion Problems (Distance Problems)

Recall that the distance traveled is equal to the average rate times the time traveled at that rate, $D=r\cdot t$.
These uniform motion problems usually have a lot of data, so it helps to first organize that data in a chart and then set up a linear system. In this section, you are encouraged to use two variables.

Example 7: An executive traveled a total of 8 hours and 1,930 miles by car and by plane. Driving to the airport by car, she averaged 60 miles per hour. In the air, the plane averaged 350 miles per hour. How long did it take her to drive to the airport?

Solution: We are asked to find the time it takes her to drive to the airport; this indicates that time is the unknown quantity.

Let x represent the time it took to drive to the airport.
Let y represent the time spent in the air.

	Distance =	***Rate*** ×	***Time***
Travel by car		60 mph	x
Travel by air		350 mph	y
Total	1,930 mi		8 hours

Use the formula D=r·t to fill in the unknown distances.

$$\text{Distance traveled in the car}: \quad D = r \cdot t = 60 \cdot x$$
$$\text{Distance traveled in the air}: \quad D = r \cdot t = 350 \cdot y$$

	Distance =	***Rate*** ×	***Time***
Travel by car	$60x$	60 mph	x
Travel by air	$350y$	350 mph	y
Total	1,930 mi		8 hours

The distance column and the time column of the chart help us to set up the following linear system.

	Distance =	***Rate*** ×	***Time***
Travel by car	$60x$	60 mph	x
Travel by air	$350y$	350 mph	y
Total	1,930 mi		8 hours

$$60x + 350y = 1{,}930 \qquad x + y = 8$$

$$\begin{cases} x + y = 8 & \leftarrow \textit{total time traveled} \\ 60x + 350y = 1{,}930 & \leftarrow \textit{total distance traveled} \end{cases}$$

Solve.

$$\begin{cases} x + y = 8 \\ 60x + 350y = 1{,}930 \end{cases} \overset{\times(-60)}{\Rightarrow} \begin{cases} -60x - 60y = -480 \\ 60x + 350y = 1{,}930 \end{cases}$$

$$\begin{array}{rr} & -60x - 60y = -480 \\ + & 60x + 350y = 1{,}930 \\ \hline & 290y = 1450 \\ & \dfrac{290y}{290} = \dfrac{1450}{290} \\ & y = 5 \end{array}$$

Now back substitute to find the time it took to drive to the airport *x*:

$$\begin{aligned} x + y &= 8 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$

Answer: It took her 3 hours to drive to the airport.

It is not always the case that time is the unknown quantity. Read the problem carefully and identify what you are asked to find; this defines your variables.

Example 8: Flying with the wind, an airplane traveled 1,365 miles in 3 hours. The plane then turned against the wind and traveled another 870 miles in 2 hours. Find the speed of the airplane and the speed of the wind.

Solution: There is no obvious relationship between the speed of the plane and the speed of the wind. For this reason, use two variables as follows:

Let x represent the speed of the airplane.
Let w represent the speed of the wind.

Use the following chart to organize the data:

	Distance =	***Rate*** ×	***Time***
Flight with wind	1,365 mi		3 hrs
Flight against wind	870 mi		2 hrs
Total			

With the wind, the airplane's total speed is x+w. Flying against the wind, the total speed is x−w.

	Distance =	Rate ×	Time
Flight with wind	1,365 mi	$x + w$	3 hrs
Flight against wind	870 mi	$x - w$	2 hrs
Total	2,235 mi		5 hrs

Use the rows of the chart along with the formula D=r·t to construct a linear system that models this problem. Take care to group the quantities that represent the rate in parentheses.

	Distance =	Rate ×	Time
Flight with wind	1,365 mi	$x + w$	3 hrs
Flight against wind	870 mi	$x - w$	2 hrs
Total	2,235 mi		5 hrs

$$1{,}365 = (x + w)\cdot 3$$
$$870 = (x - w)\cdot 2$$

$$\begin{cases} 1{,}365 = (x + w)\cdot 3 & \leftarrow \textit{distance traveled with the wind} \\ 870 = (x - w)\cdot 2 & \leftarrow \textit{distance traveled against the wind} \end{cases}$$

If we divide both sides of the first equation by 3 and both sides of the second equation by 2, then we obtain the following equivalent system:

$$\begin{cases} 1{,}365 = (x + w)\cdot 3 \\ 870 = (x - w)\cdot 2 \end{cases} \begin{matrix} \overset{\div 3}{\Rightarrow} \\ \underset{\div 2}{\Rightarrow} \end{matrix} \begin{cases} 455 = x + w \\ 435 = x - w \end{cases}$$

$$\begin{array}{rr} & x + w = 455 \\ + & x - w = 435 \\ \hline & 2x \quad = 890 \\ & \dfrac{2x}{2} = \dfrac{890}{2} \\ & x = 445 \end{array}$$

Back substitute.

$$\begin{aligned} x + w &= 455 \\ 445 + w &= 455 \\ w &= 10 \end{aligned}$$

Answer: The speed of the airplane is 445 miles per hour and the speed of the wind is 10 miles per hour.

Try this! A boat traveled 24 miles downstream in 2 hours. The return trip, which was against the current, took twice as long. What are the speeds of the boat and of the current?

Answer: The speed of the boat is 9 miles per hour and the speed of the current is 3 miles per hour.

Video Solution (click to see video)

4.6 Practice – Applications of Systems

Set up a linear system and solve.

1. The sum of two integers is 54 and their difference is 10. Find the integers.

2. The sum of two integers is 50 and their difference is 24. Find the integers.

3. The sum of two positive integers is 32. When the smaller integer is subtracted from twice the larger, the result is 40. Find the two integers.

4. The sum of two positive integers is 48. When twice the smaller integer is subtracted from the larger, the result is 12. Find the two integers.

5. The sum of two integers is 74. The larger is 26 more than twice the smaller. Find the two integers.

6. The sum of two integers is 45. The larger is 3 less than three times the smaller. Find the two integers.

7. The sum of two numbers is zero. When 4 times the smaller number is added to 8 times the larger, the result is 1. Find the two numbers.

8. The sum of a larger number and 4 times a smaller number is 5. When 8 times the smaller is subtracted from twice the larger, the result is −2. Find the numbers.

9. The sum of 12 times the larger number and 11 times the smaller is −36. The difference of 12 times the larger and 7 times the smaller is 36. Find the numbers.

10. The sum of 4 times the larger number and 3 times the smaller is 7. The difference of 8 times the larger and 6 times the smaller is 10. Find the numbers.

Part B: Interest and Money Problems

Set up a linear system and solve.

11. A $7,000 principal is invested in two accounts, one earning 3% interest and another earning 7% interest. If the total interest for the year is $262, then how much is invested in each account?

12. Mary has her total savings of $12,500 in two different CD accounts. One CD earns 4.4% interest and another earns 3.2% interest. If her total interest for the year is $463, then how much does she have in each CD account?

13. Sally's $1,800 savings is in two accounts. One account earns 6% annual interest and the other earns 3%. Her total interest for the year is $93. How much does she have in each account?

14. Joe has two savings accounts totaling $4,500. One account earns 334% annual interest and the other earns 258%. If his total interest for the year is $141.75, then how much is in each account?

15. Millicent has $10,000 invested in two accounts. For the year, she earns $535 more in interest from her 7% mutual fund account than she does from her 4% CD. How much does she have in each account?

16. A small business has $85,000 invested in two accounts. If the account earning 3% annual interest earns $825 more in interest than the account earning 4.5% annual interest, then how much is invested in each account?

17. Jerry earned a total of $284 in simple interest from two separate accounts. In an account earning 6% interest, Jerry invested $1,000 more than twice the amount he invested in an account earning 4%. How much did he invest in each account?

18. James earned a total of $68.25 in simple interest from two separate accounts. In an account earning 2.6% interest, James invested one-half as much as he did in the other account that earned 5.2%. How much did he invest in each account?

19. A cash register contains $10 bills and $20 bills with a total value of $340. If there are 23 bills total, then how many of each does the register contain?

20. John was able to purchase a pizza for $10.80 with quarters and dimes. If he uses 60 coins to buy the pizza, then how many of each did he have?

21. Dennis mowed his neighbor's lawn for a jar of dimes and nickels. Upon completing the job, he counted the coins and found that there were 4 less than twice as many dimes as there were nickels. The total value of all the coins is $6.60. How many of each coin did he have?

22. Two families bought tickets for the big football game. One family ordered 2 adult tickets and 3 children's tickets for a total of $26.00. Another family ordered 3 adult tickets and 4 children's tickets for a total of $37.00. How much did each adult ticket cost?

23. Two friends found shirts and shorts on sale at a flea market. One bought 5 shirts and 3 shorts for a total of $51.00. The other bought 3 shirts and 7 shorts for a total of $80.00. How much was each shirt and each pair of shorts?

24. On Monday Joe bought 10 cups of coffee and 5 doughnuts for his office at a cost of $16.50. It turns out that the doughnuts were more popular than the coffee. Therefore, on Tuesday he bought 5 cups of coffee and 10 doughnuts for a total of $14.25. How much was each cup of coffee?

Part C: Mixture Problems

Set up a linear system and solve.

25. A 15% acid solution is to be mixed with a 25% acid solution to produce 12 gallons of a 20% acid solution. How much of each is needed?

26. One alcohol solution contains 12% alcohol and another contains 26% alcohol. How much of each should be mixed together to obtain 5 gallons of a 14.8% alcohol solution?

27. A nurse wishes to obtain 40 ounces of a 1.2% saline solution. How much of a 1% saline solution must she mix with a 2% saline solution to achieve the desired result?

28. A customer ordered 20 pounds of fertilizer that contains 15% nitrogen. To fill the customer's order, how much of the stock 30% nitrogen fertilizer must be mixed with the 10% nitrogen fertilizer?

29. A customer ordered 2 pounds of a mixed peanut product containing 15% cashews. The inventory consists of only two mixes containing 10% and 30% cashews. How much of each type must be mixed to fill the order?

30. How many pounds of pure peanuts must be combined with a 20% peanut mix to produce 10 pounds of a 32% peanut mix?

31. How much cleaning fluid with 20% alcohol content, must be mixed with water to obtain a 24-ounce mixture with 10% alcohol content?

32. A chemist wishes to create a 32-ounce solution with 12% acid content. He uses two types of stock solutions, one with 30% acid content and another with 10% acid content. How much of each does he need?

33. A concentrated cleaning solution that contains 50% ammonia is mixed with another solution containing 10% ammonia. How much of each is mixed to obtain 8 ounces of a 32% ammonia cleaning formula?

34. A 50% fruit juice concentrate can be purchased wholesale. Best taste is achieved when water is mixed with the concentrate in such a way as to obtain a 12% fruit juice mixture. How much water and concentrate is needed to make a 50-ounce fruit juice drink?

35. A 75% antifreeze concentrate is to be mixed with water to obtain 6 gallons of a 25% antifreeze solution. How much water is needed?

36. Pure sugar is to be mixed with a fruit salad containing 10% sugar to produce 48 ounces of a salad containing 16% sugar. How much pure sugar is required?

Part D: Uniform Motion Problems

Set up a linear system and solve

37. An airplane averaged 460 miles per hour on a trip with the wind behind it and 345 miles per hour on the return trip against the wind. If the total round trip took 7 hours, then how long did the airplane spend on each leg of the trip?

38. The two legs of a 330-mile trip took 5 hours. The average speed for the first leg of the trip was 70 miles per hour and the average speed for the second leg of the trip was 60 miles per hour. How long did each leg of the trip take?

39. An executive traveled 1,200 miles, part by helicopter and part by private jet. The jet averaged 320 miles per hour while the helicopter averaged 80 miles per hour. If the total trip took 4½ hours, then how long did she spend in the private jet?

40. Joe took two buses on the 463-mile trip from San Jose to San Diego. The first bus averaged 50 miles per hour and the second bus was able to average 64 miles per hour. If the total trip took 8 hours, then how long was spent in each bus?

41. Billy canoed downstream to the general store at an average rate of 9 miles per hour. His average rate canoeing back upstream was 4 miles per hour. If the total trip took 6½ hours, then how long did it take Billy to get back on the return trip?

42. Two brothers drove the 2,793 miles from Los Angeles to New York. One of the brothers, driving in the day, was able to average 70 miles per hour, and the other, driving at night, was able to average 53 miles per hour. If the total trip took 45 hours, then how many hours did each brother drive?

43. A boat traveled 24 miles downstream in 2 hours. The return trip took twice as long. What was the speed of the boat and the current?

44. A helicopter flying with the wind can travel 525 miles in 5 hours. On the return trip, against the wind, it will take 7 hours. What are the speeds of the helicopter and of the wind?

45. A boat can travel 42 miles with the current downstream in 3 hours. Returning upstream against the current, the boat can only travel 33 miles in 3 hours. Find the speed of the current.

46. A light aircraft flying with the wind can travel 180 miles in 1½ hours. The aircraft can fly the same distance against the wind in 2 hours. Find the speed of the wind.

Part E: Discussion Board

47. Compose a number or money problem that can be solved with a system of equations of your own and share it on the discussion board.

48. Compose a mixture problem that can be solved with a system of equations of your own and share it on the discussion board.

49. Compose a uniform motion problem that can be solved with a system of equations of your own and share it on the discussion board.

Chapter 5 : Polynomials

5.1

Polynomials - Exponent Properties

Objective: Simplify expressions using the properties of exponents.

Problems with expoenents can often be simplified using a few basic exponent properties. Exponents represent repeated multiplication. We will use this fact to discover the important properties.

World View Note: The word exponent comes from the Latin "expo" meaning out of and "ponere" meaning place. While there is some debate, it seems that the Babylonians living in Iraq were the first to do work with exponents (dating back to the 23rd century BC or earlier!)

Example 196.

$$\begin{array}{rl} a^3a^2 & \text{Expand exponents to multiplication problem} \\ (aaa)(aa) & \text{Now we have } 5\, a's \text{ being multiplied together} \\ a^5 & \text{Our Solution} \end{array}$$

A quicker method to arrive at our answer would have been to just add the exponents: $a^3a^2 = a^{3+2} = a^5$ This is known as the **product rule of exponents**

$$\textbf{Product Rule of Exponents: } \boldsymbol{a^m a^n = a^{m+n}}$$

The product rule of exponents can be used to simplify many problems. We will add the exponent on like variables. This is shown in the following examples

Example 197.

$$\begin{array}{rl} 3^2\cdot 3^6\cdot 3 & \text{Same base, add the exponents } 2+6+1 \\ 3^9 & \text{Our Solution} \end{array}$$

Example 198.

$$\begin{array}{rl} 2x^3y^5z\cdot 5xy^2z^3 & \text{Multiply } 2\cdot 5, \text{add exponents on } x, y \text{ and } z \\ 10x^4y^7z^4 & \text{Our Solution} \end{array}$$

Rather than multiplying, we will now try to divide with exponents

Example 199.

$$\begin{array}{rl} \dfrac{a^5}{a^2} & \text{Expand exponents} \\ \dfrac{aaaaa}{aa} & \text{Divide out two of the } a's \\ aaa & \text{Convert to exponents} \\ a^3 & \text{Our Solution} \end{array}$$

A quicker method to arrive at the solution would have been to just subtract the exponents, $\frac{a^5}{a^2} = a^{5-2} = a^3$. This is known as the quotient rule of exponents.

$$\textbf{Quotient Rule of Exponents: } \boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$$

The quotient rule of exponents can similarly be used to simplify exponent problems by subtracting exponents on like variables. This is shown in the following examples.

Example 200.

$$\begin{array}{rl} \dfrac{7^{13}}{7^5} & \text{Same base, subtract the exponents} \\ 7^8 & \text{Our Solution} \end{array}$$

Example 201.

$$\begin{array}{rl} \dfrac{5a^3b^5c^2}{2\mathrm{ab}^3c} & \text{Subtract exponents on } a, b \text{ and } c \\ \\ \dfrac{5}{2}a^2b^2c & \text{Our Solution} \end{array}$$

A third property we will look at will have an exponent problem raised to a second exponent. This is investigated in the following example.

Example 202.

$$\begin{array}{rl} \left(a^2\right)^3 & \text{This means we have } a^2 \text{ three times} \\ a^2 \cdot a^2 \cdot a^2 & \text{Add exponents} \\ a^6 & \text{Our solution} \end{array}$$

A quicker method to arrive at the solution would have been to just multiply the exponents, $(a^2)^3 = a^{2 \cdot 3} = a^6$. This is known as the power of a power rule of exponents.

$$\textbf{Power of } \boldsymbol{a} \textbf{ Power Rule of Exponents: } \boldsymbol{(a^m)^n = a^{mn}}$$

This property is often combined with two other properties which we will investigate now.

Example 203.

$$\begin{array}{rl} (ab)^3 & \text{This means we have } (ab) \text{ three times} \\ (ab)(ab)(ab) & \text{Three } a's \text{ and three } b's \text{ can be written with exponents} \\ a^3b^3 & \text{Our Solution} \end{array}$$

A quicker method to arrive at the solution would have been to take the exponent of three and put it on each factor in parenthesis, $(ab)^3 = a^3b^3$. This is known as the power of a product rule or exponents.

$$\textbf{Power of } \boldsymbol{a} \textbf{ Product Rule of Exponents: } (\mathbf{ab})^{\boldsymbol{m}} = \boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{m}}$$

It is important to be careful to only use the power of a product rule with multiplication inside parenthesis. This property does NOT work if there is addition or subtraction.

Warning 204.

$$(a+b)^m \neq a^m + b^m \qquad \text{These are } \textbf{NOT} \text{ equal, beware of this error!}$$

Another property that is very similar to the power of a product rule is considered next.

Example 205.

$$\left(\frac{a}{b}\right)^3 \qquad \text{This means we have the fraction three timse}$$

$$\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) \qquad \text{Multiply fractions across the top and bottom, using exponents}$$

$$\frac{a^3}{b^3} \qquad \text{Our Solution}$$

A quicker method to arrive at the solution would have been to put the exponent on every factor in both the numerator and denominator, $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$. This is known as the power of a quotient rule of exponents.

$$\textbf{Power of } \boldsymbol{a} \textbf{ Quotient Rule of Exponents: } \left(\frac{\boldsymbol{a}}{\boldsymbol{b}}\right)^{\boldsymbol{m}} = \frac{\boldsymbol{a}^{\boldsymbol{m}}}{\boldsymbol{b}^{\boldsymbol{m}}}$$

The power of a power, product and quotient rules are often used together to simplify expressions. This is shown in the following examples.

Example 206.

$$(x^3yz^2)^4 \qquad \text{Put the exponent of 4 on each factor, multiplying powers}$$

$$x^{12}y^4z^8 \qquad \text{Our solution}$$

Example 207.

$$\left(\frac{a^3b}{c^8d^5}\right)^2 \quad \text{Put the exponent of 2 on each factor, multiplying powers}$$

$$\frac{a^6b^2}{c^8d^{10}} \quad \text{Our Solution}$$

As we multiply exponents its important to remember these properties apply to exponents, not bases. An expressions such as 5^3 does not mean we multipy 5 by 3, rather we multiply 5 three times, $5 \times 5 \times 5 = 125$. This is shown in the next example.

Example 208.

$$(4x^2y^5)^3 \quad \text{Put the exponent of 3 on each factor, multiplying powers}$$
$$4^3x^6y^{15} \quad \text{Evaluate } 4^3$$
$$64x^6y^{15} \quad \text{Our Solution}$$

In the previous example we did not put the 3 on the 4 and multipy to get 12, this would have been incorrect. Never multipy a base by the exponent. These properties pertain to exponents only, not bases.

In this lesson we have discussed 5 different exponent properties. These rules are summarized in the following table.

Rules of Exponents

Product Rule of Exponents	$a^ma^n = a^{m+n}$
Quotient Rule of Exponents	$\frac{a^m}{a^n} = a^{m-n}$
Power of *a* Power Rule of Exponents	$(a^m)^n = a^{mn}$
Power of *a* Product Rule of Exponents	$(ab)^m = a^mb^m$
Power of *a* Quotient Rule of Exponents	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

These five properties are often mixed up in the same problem. Often there is a bit of flexibility as to which property is used first. However, order of operations still applies to a problem. For this reason it is the suggestion of the auther to simplify inside any parenthesis first, then simplify any exponents (using power rules), and finally simplify any multiplication or division (using product and quotient rules). This is illustrated in the next few examples.

Example 209.

$$(4x^3y \cdot 5x^4y^2)^3 \quad \text{In parenthesis simplify using product rule, adding exponents}$$
$$(20x^7y^3)^3 \quad \text{With power rules, put three on each factor, multiplying exponents}$$
$$20^3x^{21}y^9 \quad \text{Evaluate } 20^3$$
$$8000x^{21}y^9 \quad \text{Our Solution}$$

Example 210.

$$\begin{array}{rl} 7a^3(2a^4)^3 & \text{Parenthesis are already simplified, next use power rules} \\ 7a^3(8a^{12}) & \text{Using product rule, add exponents and multiply numbers} \\ 56a^{15} & \text{Our Solution} \end{array}$$

Example 211.

$$\begin{array}{rl} \dfrac{3a^3b\cdot 10a^4b^3}{2a^4b^2} & \text{Simplify numerator with product rule, adding exponents} \\ \dfrac{30a^7b^4}{2a^4b^2} & \text{Now use the quotient rule to subtract exponents} \\ 15a^3b^2 & \text{Our Solution} \end{array}$$

Example 212.

$$\begin{array}{rl} \dfrac{3m^8n^{12}}{(m^2n^3)^3} & \text{Use power rule in denominator} \\ \dfrac{3m^8n^{12}}{m^6n^9} & \text{Use quotient rule} \\ 3m^2n^3 & \text{Our solution} \end{array}$$

Example 213.

$$\begin{array}{rl} \left(\dfrac{3\mathrm{ab}^2(2a^4b^2)^3}{6a^5b^7}\right)^2 & \text{Simplify inside parenthesis first, using power rule in numerator} \\ \left(\dfrac{3\mathrm{ab}^2(8a^{12}b^6)}{6a^5b^7}\right)^2 & \text{Simplify numerator using product rule} \\ \left(\dfrac{24a^{13}b^8}{6a^5b^7}\right)^2 & \text{Simplify using the quotient rule} \\ (4a^8b)^2 & \text{Now that the parenthesis are simplified, use the power rules} \\ 16a^{16}b^2 & \text{Our Solution} \end{array}$$

Clearly these problems can quickly become quite involved. Remember to follow order of operations as a guide, simplify inside parenthesis first, then power rules, then product and quotient rules.

5.1 Practice - Exponent Properties

Simplify.

1) $4\cdot 4^4\cdot 4^4$

2) $4\cdot 4^4\cdot 4^2$

3) $4\cdot 2^2$

4) $3\cdot 3^3\cdot 3^2$

5) $3m\cdot 4\text{mn}$

6) $3x\cdot 4x^2$

7) $2m^4n^2\cdot 4nm^2$

8) $x^2y^4\cdot xy^2$

9) $(3^3)^4$

10) $(4^3)^4$

11) $(4^4)^2$

12) $(3^2)^3$

13) $(2u^3v^2)^2$

14) $(xy)^3$

15) $(2a^4)^4$

16) $(2xy)^4$

17) $\frac{4^5}{4^3}$

18) $\frac{3^7}{3^3}$

19) $\frac{3^2}{3}$

20) $\frac{3^4}{3}$

21) $\frac{3nm^2}{3n}$

22) $\frac{x^2y^4}{4xy}$

23) $\frac{4x^3y^4}{3xy^3}$

24) $\frac{xy^3}{4xy}$

25) $(x^3y^4\cdot 2x^2y^3)^2$

26) $(u^2v^2\cdot 2u^4)^3$

27) $2x(x^4y^4)^4$

28) $\frac{3vu^5\cdot 2v^3}{uv^2\cdot 2u^3v}$

29) $\frac{2x^7y^5}{3x^3y\cdot 4x^2y^3}$

30) $\frac{2ba^7\cdot 2b^4}{ba^2\cdot 3a^3b^4}$

31) $\left(\frac{(2x)^3}{x^3}\right)^2$

32) $\frac{2a^2b^2a^7}{(ba^4)^2}$

33) $\left(\frac{2y^{17}}{(2x^2y^4)^4}\right)^3$

34) $\frac{yx^2\cdot (y^4)^2}{2y^4}$

35) $\left(\frac{2mn^4\cdot 2m^4n^4}{mn^4}\right)^3$

36) $\frac{n^3(n^4)^2}{2mn}$

37) $\frac{2xy^5\cdot 2x^2y^3}{2xy^4\cdot y^3}$

38) $\frac{(2y^3x^2)^2}{2x^2y^4\cdot x^2}$

39) $\frac{q^3r^2\cdot (2p^2q^2r^3)^2}{2p^3}$

40) $\frac{2x^4y^5\cdot 2z^{10}x^2y^7}{(xy^2z^2)^4}$

41) $\left(\frac{zy^3\cdot z^3x^4y^4}{x^3y^3z^3}\right)^4$

42) $\left(\frac{2q^3p^3r^4\cdot 2p^3}{(qrp^3)^2}\right)^4$

43) $\frac{2x^2y^2z^6\cdot 2zx^2y^2}{(x^2z^3)^2}$

5.2

Polynomials - Negative Exponents

Objective: Simplify expressions with negative exponents using the properties of exponents.

There are a few special exponent properties that deal with exponents that are not positive. The first is considered in the following example, which is worded out 2 different ways:

Example 214.

$\frac{a^3}{a^3}$	Use the quotient rule to subtract exponents
a^0	Our Solution, but now we consider the problem a the second way:
$\frac{a^3}{a^3}$	Rewrite exponents as repeated multiplication
$\frac{aaa}{aaa}$	Reduce out all the $a's$
$\frac{1}{1}=1$	Our Solution, when we combine the two solutions we get:
$a^0=1$	Our final result.

This final result is an imporant property known as the zero power rule of exponents

Zero Power Rule of Exponents: $a^0 = 1$

Any number or expression raised to the zero power will always be 1. This is illustrated in the following example.

Example 215.

$(3x^2)^0$	Zero power rule
1	Our Solution

Another property we will consider here deals with negative exponents. Again we will solve the following example two ways.

Example 216.

$$\frac{a^3}{a^5} \quad \text{Using the quotient rule, subtract exponents}$$

$$a^{-2} \quad \text{Our Solution, but we will also solve this problem another way.}$$

$$\frac{a^3}{a^5} \quad \text{Rewrite exponents as repeated multiplication}$$

$$\frac{aaa}{aaaaa} \quad \text{Reduce three } a's \text{ out of top and bottom}$$

$$\frac{1}{aa} \quad \text{Simplify to exponents}$$

$$\frac{1}{a^2} \quad \text{Our Solution, putting these solutions together gives:}$$

$$a^{-2} = \frac{1}{a^2} \quad \text{Our Final Solution}$$

This example illustrates an important property of exponents. Negative exponents yield the reciprocal of the base. Once we take the reciprical the exponent is now positive. Also, it is important to note a negative exponent does not mean the expression is negative, only that we need the reciprocal of the base. Following are the rules of negative exponents

Rules of Negative Exponets:

$$\boldsymbol{a^{-m} = \frac{1}{m}}$$

$$\boldsymbol{\frac{1}{a^{-m}} = a^m}$$

$$\boldsymbol{\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}}$$

Negative exponents can be combined in several different ways. As a general rule if we think of our expression as a fraction, negative exponents in the numerator must be moved to the denominator, likewise, negative exponents in the denominator need to be moved to the numerator. When the base with exponent moves, the exponent is now positive. This is illustrated in the following example.

Example 217.

$$\frac{a^3b^{-2}c}{2d^{-1}e^{-4}f^2} \quad \text{Negative exponents on } b, d, \text{ and } e \text{ need to flip}$$

$$\frac{a^3cde^4}{2b^2f^2} \quad \text{Our Solution}$$

As we simplified our fraction we took special care to move the bases that had a negative exponent, but the expression itself did not become negative because of those exponents. Also, it is important to remember that exponents only effect what they are attached to. The 2 in the denominator of the above example does not have an exponent on it, so it does not move with the d.

We now have the following nine properties of exponents. It is important that we are very familiar with all of them.

Properties of Exponents

$\boldsymbol{a^m a^n = a^{m+n}}$	$\boldsymbol{(ab)^m = a^m b^m}$	$\boldsymbol{a^{-m} = \frac{1}{a^m}}$
$\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$	$\boldsymbol{\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}}$	$\boldsymbol{\frac{1}{a^{-m}} = a^m}$
$\boldsymbol{(a^m)^n = a^{mn}}$	$\boldsymbol{a^0 = 1}$	$\boldsymbol{\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}}$

World View Note: Nicolas Chuquet, the French mathematician of the 15th century wrote $12^{1\bar{m}}$ to indicate $12x^{-1}$. This was the first known use of the negative exponent.

Simplifying with negative exponents is much the same as simplifying with positive exponents. It is the advice of the author to keep the negative exponents until the end of the problem and then move them around to their correct location (numerator or denominator). As we do this it is important to be very careful of rules for adding, subtracting, and multiplying with negatives. This is illustrated in the following examples

Example 218.

$\frac{4x^{-5}y^{-3}\cdot 3x^3y^{-2}}{6x^{-5}y^3}$	Simplify numerator with product rule, adding exponents
$\frac{12x^{-2}y^{-5}}{6x^{-5}y^3}$	Quotient rule to subtract exponets, be careful with negatives!
	$(-2)-(-5)=(-2)+5=3$
	$(-5)-3=(-5)+(-3)=-8$
$2x^3y^{-8}$	Negative exponent needs to move down to denominator
$\frac{2x^3}{y^8}$	Our Solution

Example 219.

$\dfrac{(3\mathrm{ab}^3)^{-2}\mathrm{ab}^{-3}}{2a^{-4}b^0}$ In numerator, use power rule with -2, multiplying exponents
In denominator, $b^0=1$

$\dfrac{3^{-2}a^{-2}b^{-6}\mathrm{ab}^{-3}}{2a^{-4}}$ In numerator, use product rule to add exponents

$\dfrac{3^{-2}a^{-1}b^{-9}}{2a^{-4}}$ Use quotient rule to subtract exponents, be careful with negatives
$(-1)-(-4)=(-1)+4=3$

$\dfrac{3^{-2}a^3b^{-9}}{2}$ Move 3 and b to denominator because of negative exponents

$\dfrac{a^3}{3^2 2b^9}$ Evaluate $3^2 2$

$\dfrac{a^3}{18b^9}$ Our Solution

In the previous example it is important to point out that when we simplified 3^{-2} we moved the three to the denominator and the exponent became positive. We did not make the number negative! Negative exponents never make the bases negative, they simply mean we have to take the reciprocal of the base. One final example with negative exponents is given here.

Example 220.

$\left(\dfrac{3x^{-2}y^5z^3\cdot 6x^{-6}y^{-2}z^{-3}}{9(x^2y^{-2})^{-3}}\right)^{-3}$ In numerator, use product rule, adding exponents
In denominator, use power rule, multiplying exponets

$\left(\dfrac{18x^{-8}y^3z^0}{9x^{-6}y^6}\right)^{-3}$ Use quotient rule to subtract exponents,

be careful with negatives:
$(-8)-(-6)=(-8)+6=-2$
$3-6=3+(-6)=-3$

$(2x^{-2}y^{-3}z^0)^{-3}$ Parenthesis are done, use power rule with -3

$2^{-3}x^6y^9z^0$ Move 2 with negative exponent down and $z^0=1$

$\dfrac{x^6y^9}{2^3}$ Evaluate 2^3

$\dfrac{x^6y^9}{8}$ Our Solution

5.2 Practice - Negative Exponents

Simplify. Your answer should contain only positive expontents.

1) $2x^4y^{-2}\cdot(2xy^3)^4$

2) $2a^{-2}b^{-3}\cdot(2a^0b^4)^4$

3) $(a^4b^{-3})^3\cdot 2a^3b^{-2}$

4) $2x^3y^2\cdot(2x^3)^0$

5) $(2x^2y^2)^4x^{-4}$

6) $(m^0n^3\cdot 2m^{-3}n^{-3})^0$

7) $(x^3y^4)^3\cdot x^{-4}y^4$

8) $2m^{-1}n^{-3}\cdot(2m^{-1}n^{-3})^4$

9) $\frac{2x^{-3}y^2}{3x^{-3}y^3\cdot 3x^0}$

10) $\frac{3y^3}{3yx^3\cdot 2x^4y^{-3}}$

11) $\frac{4xy^{-3}\cdot x^{-4}y^0}{4y^{-1}}$

12) $\frac{3x^3y^2}{4y^{-2}\cdot 3x^{-2}y^{-4}}$

13) $\frac{u^2v^{-1}}{2u^0v^4\cdot 2uv}$

14) $\frac{2xy^2\cdot 4x^3y^{-4}}{4x^{-4}y^{-4}\cdot 4x}$

15) $\frac{u^2}{4u^0v^3\cdot 3v^2}$

16) $\frac{2x^{-2}y^2}{4yx^2}$

17) $\frac{2y}{(x^0y^2)^4}$

18) $\frac{(a^4)^4}{2b}$

19) $(\frac{2a^2b^3}{a^{-1}})^4$

20) $(\frac{2y^{-4}}{x^2})^{-2}$

21) $\frac{2nm^4}{(2m^2n^2)^4}$

22) $\frac{2y^2}{(x^4y^0)^{-4}}$

23) $\frac{(2mn)^4}{m^0n^{-2}}$

24) $\frac{2x^{-3}}{(x^4y^{-3})^{-1}}$

25) $\frac{y^3\cdot x^{-3}y^2}{(x^4y^2)^3}$

26) $\frac{2x^{-2}y^0\cdot 2xy^4}{(xy^0)^{-1}}$

27) $\frac{2u^{-2}v^3\cdot(2uv^4)^{-1}}{2u^{-4}v^0}$

28) $\frac{2yx^2\cdot x^{-2}}{(2x^0y^4)^{-1}}$

29) $(\frac{2x^0\cdot y^4}{y^4})^3$

30) $\frac{u^{-3}v^{-4}}{2v(2u^{-3}v^4)^0}$

31) $\frac{y(2x^4y^2)^2}{2x^4y^0}$

32) $\frac{b^{-1}}{(2a^4b^0)^0\cdot 2a^{-3}b^2}$

33) $\frac{2yzx^2}{2x^4y^4z^{-2}\cdot(zy^2)^4}$

34) $\frac{2b^4c^{-2}\cdot(2b^3c^2)^{-4}}{a^{-2}b^4}$

35) $\frac{2kh^0\cdot 2h^{-3}k^0}{(2kj^3)^2}$

36) $(\frac{(2x^{-3}y^0z^{-1})^3\cdot x^{-3}y^2}{2x^3})^{-2}$

37) $\frac{(cb^3)^2\cdot 2a^{-3}b^2}{(a^3b^{-2}c^3)^3}$

38) $\frac{2q^4\cdot m^2p^2q^4}{(2m^{-4}p^2)^3}$

39) $\frac{(yx^{-4}z^2)^{-1}}{z^3\cdot x^2y^3z^{-1}}$

40) $\frac{2mpn^{-3}}{(m^0n^{-4}p^2)^3\cdot 2n^2p^0}$

5.3

Polynomials - Scientific Notation

Objective: Multiply and divide expressions using scientific notation and exponent properties.

One application of exponent properties comes from scientific notation. Scientific notation is used to represent really large or really small numbers. An example of really large numbers would be the distance that light travels in a year in miles. An example of really small numbers would be the mass of a single hydrogen atom in grams. Doing basic operations such as multiplication and division with these numbers would normally be very combersome. However, our exponent properties make this process much simpler.

First we will take a look at what scientific notation is. Scientific notation has two parts, a number between one and ten (it can be equal to one, but not ten), and that number multiplied by ten to some exponent.

Scientific Notation: $\boldsymbol{a \times 10^b}$ **where** $\boldsymbol{1 \leqslant a < 10}$

The exponent, b, is very important to how we convert between scientific notation and normal numbers, or standard notation. The exponent tells us how many times we will multiply by 10. Multiplying by 10 in affect moves the decimal point one place. So the exponent will tell us how many times the exponent moves between scientific notation and standard notation. To decide which direction to move the decimal (left or right) we simply need to remember that positive exponents mean in standard notation we have a big number (bigger than ten) and negative exponents mean in standard notation we have a small number (less than one).

Keeping this in mind, we can easily make conversions between standard notation and scientific notation.

Example 221.

Convert $14,200$ to scientific notation	Put decimal after first nonzero number
1.42	Exponent is how many times decimal moved, 4
$\times 10^4$	Positive exponent, standard notation is big
1.42×10^4	Our Solution

Example 222.

Convert 0.0042 to scientific notation	Put decimal after first nonzero number
4.2	Exponent is how many times decimal moved, 3
$\times 10^{-3}$	Negative exponent, standard notation is small
4.2×10^{-3}	Our Solution

Example 223.

$$\begin{array}{rl} \text{Convert } 3.21\times 10^5 \text{ to standard notation} & \text{Positive exponent means standard notation} \\ & \text{big number. Move decimal right 5 places} \\ 321,000 & \text{Our Solution} \end{array}$$

Example 224.

$$\begin{array}{rl} \text{Conver } 7.4\times 10^{-3} \text{ to standard notation} & \text{Negative exponent means standard notation} \\ & \text{is } a \text{ small number. Move decimal left 3 places} \\ 0.0074 & \text{Our Solution} \end{array}$$

Converting between standard notation and scientific notation is important to understand how scientific notation works and what it does. Here our main interest is to be able to multiply and divide numbers in scientific notation using exponent properties. The way we do this is first do the operation with the front number (multiply or divide) then use exponent properties to simplify the 10's. Scientific notation is the only time where it will be allowed to have negative exponents in our final solution. The negative exponent simply informs us that we are dealing with small numbers. Consider the following examples.

Example 225.

$$\begin{array}{rl} (2.1\times 10^{-7})(3.7\times 10^5) & \text{Deal with numbers and } 10's \text{ separately} \\ (2.1)(3.7)=7.77 & \text{Multiply numbers} \\ 10^{-7}10^5=10^{-2} & \text{Use product rule on } 10's \text{ and add exponents} \\ 7.77\times 10^{-2} & \text{Our Solution} \end{array}$$

Example 226.

$$\begin{array}{rl} \dfrac{4.96\times 10^4}{3.1\times 10^{-3}} & \text{Deal with numbers and } 10's \text{ separately} \\ \dfrac{4.96}{3.1}=1.6 & \text{Divide Numbers} \\ \dfrac{10^4}{10^{-3}}=10^7 & \text{Use quotient rule to subtract exponents, be careful with negatives!} \\ & \text{Be careful with negatives, } 4-(-3)=4+3=7 \\ 1.6\times 10^7 & \text{Our Solution} \end{array}$$

Example 227.

$(1.8\times10^{-4})^3$	Use power rule to deal with numbers and $10's$ separately
$1.8^3=5.832$	Evaluate 1.8^3
$(10^{-4})^3=10^{-12}$	Multiply exponents
5.832×10^{-12}	Our Solution

Often when we multiply or divide in scientific notation the end result is not in scientific notation. We will then have to convert the front number into scientific notation and then combine the 10's using the product property of exponents and adding the exponents. This is shown in the following examples.

Example 228.

$(4.7\times10^{-3})(6.1\times10^9)$	Deal with numbers and $10's$ separately
$(4.7)(6.1)=28.67$	Multiply numbers
2.867×10^1	Convert this number into scientific notation
$\mathbf{10^1}10^{-3}10^9=10^7$	Use product rule, add exponents, using 10^1 from conversion
2.867×10^7	Our Solution

World View Note: Archimedes (287 BC - 212 BC), the Greek mathematician, developed a system for representing large numbers using a system very similar to scientific notation. He used his system to calculate the number of grains of sand it would take to fill the universe. His conclusion was 10^{63} grains of sand because he figured the universe to have a diameter of 10^{14} stadia or about 2 light years.

Example 229.

$\dfrac{2.014\times10^{-3}}{3.8\times10^{-7}}$	Deal with numbers and $10's$ separately
$\dfrac{2.014}{3.8}=0.53$	Divide numbers
$0.53=5.3\times10^{-1}$	Change this number into scientific notation
$\dfrac{\mathbf{10^{-1}}10^{-3}}{10^{-7}}=10^3$	Use product and quotient rule, using 10^{-1} from the conversion
	Be careful with signs:
	$(-1)+(-3)-(-7)=(-1)+(-3)+7=3$
5.3×10^3	Our Solution

5.3 Practice - Scientific Notation

Write each number in scientific notiation

1) 885
2) 0.000744
3) 0.081
4) 1.09
5) 0.039
6) 15000

Write each number in standard notation

7) $8.7 \text{ x } 10^5$
8) $2.56 \text{ x } 10^2$
9) $9 \text{ x } 10^{-4}$
10) $5 \text{ x } 10^4$
11) $2 \text{ x } 10^0$
12) $6 \text{ x } 10^{-5}$

Simplify. Write each answer in scientific notation.

13) $(7 \text{ x } 10^{-1})(2 \text{ x } 10^{-3})$
14) $(2 \times 10^{-6})(8.8 \times 10^{-5})$
15) $(5.26 \text{ x } 10^{-5})(3.16 \text{ x } 10^{-2})$
16) $(5.1 \times 10^6)(9.84 \times 10^{-1})$
17) $(2.6 \text{ x } 10^{-2})(6 \text{ x } 10^{-2})$
18) $\frac{7.4 \times 10^4}{1.7 \times 10^{-4}}$
19) $\frac{4.9 \times 10^1}{2.7 \times 10^{-3}}$
20) $\frac{7.2 \times 10^{-1}}{7.32 \times 10^{-1}}$
21) $\frac{5.33 \times 10^{-6}}{9.62 \times 10^{-2}}$
22) $\frac{3.2 \times 10^{-3}}{5.02 \times 10^0}$
23) $(5.5 \times 10^{-5})^2$
24) $(9.6 \times 10^3)^{-4}$
25) $(7.8 \times 10^{-2})^5$
26) $(5.4 \times 10^6)^{-3}$
27) $(8.03 \times 10^4)^{-4}$
28) $(6.88 \times 10^{-4})(4.23 \times 10^1)$
29) $\frac{6.1 \times 10^{-6}}{5.1 \times 10^{-4}}$
30) $\frac{8.4 \times 10^5}{7 \times 10^{-2}}$
31) $(3.6 \times 10^0)(6.1 \times 10^{-3})$
32) $(3.15 \times 10^3)(8 \times 10^{-1})$
33) $(1.8 \times 10^{-5})^{-3}$
34) $\frac{9.58 \times 10^{-2}}{1.14 \times 10^{-3}}$
35) $\frac{9 \times 10^4}{7.83 \times 10^{-2}}$
36) $(8.3 \times 10^1)^5$
37) $\frac{3.22 \times 10^{-3}}{7 \times 10^{-6}}$
38) $\frac{5 \times 10^6}{6.69 \times 10^2}$
39) $\frac{2.4 \times 10^{-6}}{6.5 \times 10^0}$
40) $(9 \times 10^{-2})^{-3}$
41) $\frac{6 \times 10^3}{5.8 \times 10^{-3}}$
42) $(2 \times 10^4)(6 \times 10^1)$

5.4

Polynomials - Introduction to Polynomials

Objective: Evaluate, add, and subtract polynomials.

Many applications in mathematics have to do with what are called polynomials. Polynomials are made up of terms. **Terms** are a product of numbers and/or variables. For example, $5x$, $2y^2$, -5, ab^3c, and x are all terms. Terms are connected to each other by addition or subtraction. Expressions are often named based on the number of terms in them. A **monomial** has one term, such as $3x^2$. A **binomial** has two terms, such as $a^2 - b^2$. A Trinomial has three terms, such as ax^2 + bx + c. The term **polynomial** means many terms. Monomials, binomials, trinomials, and expressions with more terms all fall under the umbrella of "polynomials".

If we know what the variable in a polynomial represents we can replace the variable with the number and evaluate the polynomial as shown in the following example.

Example 230.

$$\begin{array}{rl}
2x^2-4x+6 \text{ when } x=-4 & \text{Replace variable } x \text{ with } -4\\
2(-4)^2-4(-4)+6 & \text{Exponents first}\\
2(16)-4(-4)+6 & \text{Multiplication (we can do all terms at once)}\\
32+16+6 & \text{Add}\\
54 & \text{Our Solution}
\end{array}$$

It is important to be careful with negative variables and exponents. Remember the exponent only effects the number it is physically attached to. This means $-3^2=-9$ because the exponent is only attached to the 3. Also, $(-3)^2=9$ because the exponent is attached to the parenthesis and effects everything inside. When we replace a variable with parenthesis like in the previous example, the substituted value is in parenthesis. So the $(-4)^2 = 16$ in the example. However, consider the next example.

Example 231.

$$\begin{array}{rl}
-x^2+2x+6 \text{ when } x=3 & \text{Replace variable } x \text{ with } 3\\
-(3)^2+2(3)+6 & \text{Exponent only on the 3, not negative}\\
-9+2(3)+6 & \text{Multiply}\\
-9+6+6 & \text{Add}\\
3 & \text{Our Solution}
\end{array}$$

World View Note: Ada Lovelace in 1842 described a Difference Engine that would be used to caluclate values of polynomials. Her work became the foundation for what would become the modern computer (the programming language Ada was named in her honor), more than 100 years after her death from cancer.

Generally when working with polynomials we do not know the value of the variable, so we will try and simplify instead. The simplest operation with polynomials is addition. When adding polynomials we are mearly combining like terms. Consider the following example

Example 232.

$$\begin{array}{rl} (4x^3-2x+8)+(3x^3-9x^2-11) & \text{Combine like terms } 4x^3+3x^3 \text{ and } 8-11 \\ 7x^3-9x^2-2x-3 & \text{Our Solution} \end{array}$$

Generally final answers for polynomials are written so the exponent on the variable counts down. Example 3 demonstrates this with the exponent counting down 3, 2, 1, 0 (recall $x^0=1$). Subtracting polynomials is almost as fast. One extra step comes from the minus in front of the parenthesis. When we have a negative in front of parenthesis we distribute it through, changing the signs of everything inside. The same is done for the subtraction sign.

Example 233.

$$\begin{array}{rl} (5x^2-2x+7)-(3x^2+6x-4) & \text{Distribute negative through second part} \\ 5x^2-2x+7-3x^2-6x+4 & \text{Combine like terms } 5x^2-3x^3, -2x-6x, \text{ and } 7+4 \\ 2x^2-8x+11 & \text{Our Solution} \end{array}$$

Addition and subtraction can also be combined into the same problem as shown in this final example.

Example 234.

$$\begin{array}{rl} (2x^2-4x+3)+(5x^2-6x+1)-(x^2-9x+8) & \text{Distribute negative through} \\ 2x^2-4x+3+5x^2-6x+1-x^2+9x-8 & \text{Combine like terms} \\ 6x^2-x-4 & \text{Our Solution} \end{array}$$

5.4 Practice - Introduction to Polynomials

Simplify each expression.

1) $-a^3-a^2+6a-21$ when $a=-4$

2) $n^2+3n-11$ when $n=-6$

3) $n^3-7n^2+15n-20$ when $n=2$

4) $n^3-9n^2+23n-21$ when $n=5$

5) $-5n^4-11n^3-9n^2-n-5$ when $n=-1$

6) x^4-5x^3-x+13 when $x=5$

7) $x^2+9x+23$ when $x=-3$

8) $-6x^3+41x^2-32x+11$ when $x=6$

9) $x^4-6x^3+x^2-24$ when $x=6$

10) $m^4+8m^3+14m^2+13m+5$ when $m=-6$

11) $(5p-5p^4)-(8p-8p^4)$

12) $(7m^2+5m^3)-(6m^3-5m^2)$

13) $(3n^2+n^3)-(2n^3-7n^2)$

14) $(x^2+5x^3)+(7x^2+3x^3)$

15) $(8n+n^4)-(3n-4n^4)$

16) $(3v^4+1)+(5-v^4)$

17) $(1+5p^3)-(1-8p^3)$

18) $(6x^3+5x)-(8x+6x^3)$

19) $(5n^4+6n^3)+(8-3n^3-5n^4)$

20) $(8x^2+1)-(6-x^2-x^4)$

21) $(3+b^4)+(7+2b+b^4)$

22) $(1+6r^2)+(6r^2-2-3r^4)$

23) $(8x^3+1)-(5x^4-6x^3+2)$

24) $(4n^4+6)-(4n-1-n^4)$

25) $(2a+2a^4)-(3a^2-5a^4+4a)$

26) $(6v+8v^3)+(3+4v^3-3v)$

27) $(4p^2-3-2p)-(3p^2-6p+3)$

28) $(7+4m+8m^4)-(5m^4+1+6m)$

29) $(4b^3+7b^2-3)+(8+5b^2+b^3)$

30) $(7n+1-8n^4)-(3n+7n^4+7)$

31) $(3+2n^2+4n^4)+(n^3-7n^2-4n^4)$

32) $(7x^2+2x^4+7x^3)+(6x^3-8x^4-7x^2)$

33) $(n-5n^4+7)+(n^2-7n^4-n)$

34) $(8x^2+2x^4+7x^3)+(7x^4-7x^3+2x^2)$

35) $(8r^4-5r^3+5r^2)+(2r^2+2r^3-7r^4+1)$

36) $(4x^3+x-7x^2)+(x^2-8+2x+6x^3)$

37) $(2n^2+7n^4-2)+(2+2n^3+4n^2+2n^4)$

38) $(7b^3-4b+4b^4)-(8b^3-4b^2+2b^4-8b)$

39) $(8-b+7b^3)-(3b^4+7b-8+7b^2)+(3-3b+6b^3)$

40) $(1-3n^4-8n^3)+(7n^4+2-6n^2+3n^3)+(4n^3+8n^4+7)$

41) $(8x^4+2x^3+2x)+(2x+2-2x^3-x^4)-(x^3+5x^4+8x)$

42) $(6x-5x^4-4x^2)-(2x-7x^2-4x^4-8)-(8-6x^2-4x^4)$

5.5

Polynomials - Multiplying Polynomials

Objective: Multiply polynomials.

Multiplying polynomials can take several different forms based on what we are multiplying. We will first look at multiplying monomials, then monomials by polynomials and finish with polynomials by polynomials.

Multiplying monomials is done by multiplying the numbers or coefficients and then adding the exponents on like factors. This is shown in the next example.

Example 235.

$(4x^3y^4z)(2x^2y^6z^3)$	Multiply numbers and add exponents for x, y, and z
$8x^5y^{10}z^4$	Our Solution

In the previous example it is important to remember that the z has an exponent of 1 when no exponent is written. Thus for our answer the z has an exponent of $1+3=4$. Be very careful with exponents in polynomials. If we are adding or subtracting the exponnets will stay the same, but when we multiply (or divide) the exponents will be changing.

Next we consider multiplying a monomial by a polynomial. We have seen this operation before with distributing through parenthesis. Here we will see the exact same process.

Example 236.

$4x^3(5x^2-2x+5)$	Distribute the $4x^3$, multiplying numbers, adding exponents
$20x^5-8x^4+20x^3$	Our Solution

Following is another example with more variables. When distributing the exponents on a are added and the exponents on b are added.

Example 237.

$2a^3b(3\text{ab}^2-4a)$	Distribute, multiplying numbers and adding exponents
$6a^4b^3-8a^4b$	Our Solution

There are several different methods for multiplying polynomials. All of which work, often students prefer the method they are first taught. Here three methods will be discussed. All three methods will be used to solve the same two multiplication problems.

Multiply by Distributing

Just as we distribute a monomial through parenthesis we can distribute an entire polynomial. As we do this we take each term of the second polynomial and put it in front of the first polynomial.

Example 238.

$$\begin{array}{rl}(4x+7y)(3x-2y) & \text{Distribute } (4x+7y) \text{ through parenthesis}\\ 3x(\boldsymbol{4x+7y})-2y(\boldsymbol{4x+7y}) & \text{Distribute the } 3x \text{ and } -2y\\ 12x^2+21xy-8xy-14y^2 & \text{Combine like terms } 21xy-8xy\\ 12x^2+13xy-14y^2 & \text{Our Solution}\end{array}$$

This example illustrates an important point, the negative/subtraction sign stays with the $2y$. Which means on the second step the negative is also distributed through the last set of parenthesis.

Multiplying by distributing can easily be extended to problems with more terms. First distribute the front parenthesis onto each term, then distribute again!

Example 239.

$$\begin{array}{rl}(2x-5)(4x^2-7x+3) & \text{Distribute } (2x-5) \text{ through parenthesis}\\ 4x^2(\boldsymbol{2x-5})-7x(\boldsymbol{2x-5})+3(\boldsymbol{2x-5}) & \text{Distribute again through each parenthesis}\\ 8x^3-20x^2-14x^2+35x+6x-15 & \text{Combine like terms}\\ 8x^3-34x^2+41x-15 & \text{Our Solution}\end{array}$$

This process of multiplying by distributing can easily be reversed to do an important procedure known as factoring. Factoring will be addressed in a future lesson.

Multiply by FOIL

Another form of multiplying is known as FOIL. Using the FOIL method we multiply each term in the first binomial by each term in the second binomial. The letters of FOIL help us remember every combination. F stands for First, we multiply the first term of each binomial. O stand for Outside, we multiply the outside two terms. I stands for Inside, we multiply the inside two terms. L stands for Last, we multiply the last term of each binomial. This is shown in the next example:

Example 240.

$$\begin{array}{rl}(4x+7y)(3x-2y) & \text{Use FOIL to multiply}\\ (4x)(3x)=12x^2 & F-\text{First terms } (4x)(3x)\\ (4x)(-2y)=-8xy & O-\text{Outside terms } (4x)(-2y)\\ (7y)(3x)=21xy & I-\text{Inside terms } (7y)(3x)\\ (7y)(-2y)=-14y^2 & L-\text{Last terms } (7y)(-2y)\\ 12x^2-8xy+21xy-14y^2 & \text{Combine like terms } -8xy+21xy\\ 12x^2+13xy-14y^2 & \text{Our Solution}\end{array}$$

Some students like to think of the FOIL method as distributing the first term $4x$ through the $(3x-2y)$ and distributing the second term $7y$ through the $(3x-2y)$. Thinking about FOIL in this way makes it possible to extend this method to problems with more terms.

Example 241.

$$\begin{array}{rl}(2x-5)(4x^2-7x+3) & \text{Distribute } 2x \text{ and } -5\\ (2x)(4x^2)+(2x)(-7x)+(2x)(3)-5(4x^2)-5(-7x)-5(3) & \text{Multiply out each term}\\ 8x^3-14x^2+6x-20x^2+35x-15 & \text{Combine like terms}\\ 8x^3-34x^2+41x-15 & \text{Our Solution}\end{array}$$

The second step of the FOIL method is often not written, for example, consider the previous example, a student will often go from the problem $(4x+7y)(3x-2y)$ and do the multiplication mentally to come up with $12x^2-8xy+21xy-14y^2$ and then combine like terms to come up with the final solution.

Multiplying in rows

A third method for multiplying polynomials looks very similar to multiplying numbers. Consider the problem:

$$\begin{array}{rl}35 & \\ \underline{\times 27} & \\ 245 & \text{Multiply 7 by 5 then 3}\\ \underline{700} & \text{Use 0 for placeholder, multiply 2 by 5 then 3}\\ 945 & \text{Add to get Our Solution}\end{array}$$

World View Note: The first known system that used place values comes from Chinese mathematics, dating back to 190 AD or earlier.

The same process can be done with polynomials. Multiply each term on the bottom with each term on the top.

Example 242.

$$\begin{array}{rl}(4x+7y)(3x-2y) & \text{Rewrite as vertical problem}\\ 4x+7y & \\ \underline{\times 3x-2y} & \\ -8xy-14y^2 & \text{Multiply } -2y \text{ by } 7y \text{ then } 4x\\ \underline{12x^2+21xy \qquad\quad} & \text{Multiply } 3x \text{ by } 7y \text{ then } 4x\text{. Line up like terms}\\ 12x^2+13xy-14y^2 & \text{Add like terms to get Our Solution}\end{array}$$

This same process is easily expanded to a problem with more terms.

Example 243.

$$\begin{array}{rl}
(2x-5)(4x^2-7x+3) & \text{Rewrite as vertical problem}\\
4x^3-7x+3 & \text{Put polynomial with most terms on top}\\
\underline{\times\, 2x-5} & \\
-20x^2+35x-15 & \text{Multiply } -5 \text{ by each term}\\
\underline{8x^3-14x^2+6x\qquad\quad} & \text{Multiply } 2x \text{ by each term. Line up like terms}\\
8x^3-34x^2+41x-15 & \text{Add like terms to get our solution}
\end{array}$$

This method of multiplying in rows also works with multiplying a monomial by a polynomial!

Any of the three described methods work to multiply polynomials. It is suggested that you are very comfortable with at least one of these methods as you work through the practice problems. All three methods are shown side by side in the example.

Example 244.

$$(2x-y)(4x-5y)$$

Distribute	**FOIL**	**Rows**
$4x(2x-y)-5y(2x-y)$	$2x(4x)+2x(-5y)-y(4x)-y(-5y)$	$2x-y$
$8x^2-4xy-10xy-5y^2$	$8x^2-10xy-4xy+5y^2$	$\underline{\times\, 4x-5y}$
$8x^2-14xy-5y^2$	$8x^2-14xy+5y^2$	$-10xy+5y^2$
		$\underline{8x^2-4xy\qquad\quad}$
		$8x^2-14xy+5y^2$

When we are multiplying a monomial by a polynomial by a polynomial we can solve by first multiplying the polynomials then distributing the coefficient last. This is shown in the last example.

Example 245.

$$\begin{array}{rl}
3(2x-4)(x+5) & \text{Multiply the binomials, we will use FOIL}\\
3(2x^2+10x-4x-20) & \text{Combine like terms}\\
3(2x^2+6x-20) & \text{Distribute the 3}\\
6x^2+18x-60 & \text{Our Solution}
\end{array}$$

A common error students do is distribute the three at the start into both parenthesis. While we can distribute the 3 into the $(2x-4)$ factor, distributing into both would be wrong. Be careful of this error. This is why it is suggested to multiply the binomials first, then distribute the coeffienct last.

5.5 Practice - Multiply Polynomials

Find each product.

1) $6(p-7)$

2) $4k(8k+4)$

3) $2(6x+3)$

4) $3n^2(6n+7)$

5) $5m^4(4m+4)$

6) $3(4r-7)$

7) $(4n+6)(8n+8)$

8) $(2x+1)(x-4)$

9) $(8b+3)(7b-5)$

10) $(r+8)(4r+8)$

11) $(4x+5)(2x+3)$

12) $(7n-6)(n+7)$

13) $(3v-4)(5v-2)$

14) $(6a+4)(a-8)$

15) $(6x-7)(4x+1)$

16) $(5x-6)(4x-1)$

17) $(5x+y)(6x-4y)$

18) $(2u+3v)(8u-7v)$

19) $(x+3y)(3x+4y)$

20) $(8u+6v)(5u-8v)$

21) $(7x+5y)(8x+3y)$

22) $(5a+8b)(a-3b)$

23) $(r-7)(6r^2-r+5)$

24) $(4x+8)(4x^2+3x+5)$

25) $(6n-4)(2n^2-2n+5)$

26) $(2b-3)(4b^2+4b+4)$

27) $(6x+3y)(6x^2-7xy+4y^2)$

28) $(3m-2n)(7m^2+6mn+4n^2)$

29) $(8n^2+4n+6)(6n^2-5n+6)$

30) $(2a^2+6a+3)(7a^2-6a+1)$

31) $(5k^2+3k+3)(3k^2+3k+6)$

32) $(7u^2+8uv-6v^2)(6u^2+4uv+3v^2)$

33) $3(3x-4)(2x+1)$

34) $5(x-4)(2x-3)$

35) $3(2x+1)(4x-5)$

36) $2(4x+1)(2x-6)$

37) $7(x-5)(x-2)$

38) $5(2x-1)(4x+1)$

39) $6(4x-1)(4x+1)$

40) $3(2x+3)(6x+9)$

5.6

Polynomials - Multiply Special Products

Objective: Recognize and use special product rules of a sum and difference and perfect squares to multiply polynomials.

There are a few shortcuts that we can take when multiplying polynomials. If we can recognize them the shortcuts can help us arrive at the solution much quicker. These shortcuts will also be useful to us as our study of algebra continues.

The first shortcut is often called a **sum and a difference**. A sum and a difference is easily recognized as the numbers and variables are exactly the same, but the sign in the middle is different (one sum, one difference). To illustrate the shortcut consider the following example, multiplied by the distributing method.

Example 246.

$$
\begin{array}{rl}
(a+b)(a-b) & \text{Distribute } (a+b) \\
a(a+b)-b(a+b) & \text{Distribute } a \text{ and } -b \\
a^2+ab-ab-b^2 & \text{Combine like terms } ab-ab \\
a^2-b^2 & \text{Our Solution}
\end{array}
$$

The important part of this example is the middle terms subtracted to zero. Rather than going through all this work, when we have a sum and a difference we will jump right to our solution by squaring the first term and squaring the last term, putting a subtraction between them. This is illustrated in the following example

Example 247.

$$
\begin{array}{rl}
(x-5)(x+5) & \text{Recognize sum and difference} \\
x^2-25 & \text{Square both, put subtraction between. Our Solution}
\end{array}
$$

This is much quicker than going through the work of multiplying and combining like terms. Often students ask if they can just multiply out using another method and not learn the shortcut. These shortcuts are going to be very useful when we get to factoring polynomials, or reversing the multiplication process. For this reason it is very important to be able to recognize these shortcuts. More examples are shown here.

Example 248.

$$\begin{array}{rl} (3x+7)(3x-7) & \text{Recognize sum and difference} \\ 9x^2-49 & \text{Square both, put subtraction between. Our Solution} \end{array}$$

Example 249.

$$\begin{array}{rl} (2x-6y)(2x+6y) & \text{Recognize sum and difference} \\ 4x^2-36y^2 & \text{Square both, put subtraction between. Our Solution} \end{array}$$

It is interesting to note that while we can multiply and get an answer like a^2-b^2 (with subtraction), it is impossible to multiply real numbers and end up with a product such as a^2+b^2 (with addition).

Another shortcut used to multiply is known as a **perfect square**. These are easy to recognize as we will have a binomial with a 2 in the exponent. The following example illustrates multiplying a perfect square

Example 250.

$$\begin{array}{rl} (a+b)^2 & \text{Squared is same as multiplying by itself} \\ (a+b)(a+b) & \text{Distribute } (a+b) \\ a(a+b)+b(a+b) & \text{Distribute again through final parenthesis} \\ a^2+ab+ab+b^2 & \text{Combine like terms } ab+ab \\ a^2+2\text{ab}+b^2 & \text{Our Solution} \end{array}$$

This problem also helps us find our shortcut for multiplying. The first term in the answer is the square of the first term in the problem. The middle term is 2 times the first term times the second term. The last term is the square of the last term. This can be shortened to square the first, twice the product, square the last. If we can remember this shortcut we can square any binomial. This is illustrated in the following example

Example 251.

$$\begin{array}{rl} (x-5)^2 & \text{Recognize perfect square} \\ x^2 & \text{Square the first} \\ 2(x)(-5)=-10x & \text{Twice the product} \\ (-5)^2=25 & \text{Square the last} \\ x^2-10x+25 & \text{Our Solution} \end{array}$$

Be very careful when we are squaring a binomial to **NOT** distribute the square through the parenthesis. A common error is to do the following: $(x-5)^2 = x^2 - 25$ (or $x^2 + 25$). Notice both of these are missing the middle term, $-10x$. This is why it is important to use the shortcut to help us find the correct solution. Another important observation is that the middle term in the solution always has the same sign as the middle term in the problem. This is illustrated in the next examples.

Example 252.

$$\begin{array}{rl} (2x+5)^2 & \text{Recognize perfect square} \\ (2x)^2 = 4x^2 & \text{Square the first} \\ 2(2x)(5) = 20x & \text{Twice the product} \\ 5^2 = 25 & \text{Square the last} \\ 4x^2 + 20x + 25 & \text{Our Solution} \end{array}$$

Example 253.

$$\begin{array}{rl} (3x-7y)^2 & \text{Recognize perfect square} \\ 9x^2 - 42\text{xy} + 49y^2 & \text{Square the first, twice the product, square the last. Our Solution} \end{array}$$

Example 254.

$$\begin{array}{rl} (5a+9b)^2 & \text{Recognize perfect square} \\ 25a^2 + 90\text{ab} + 81b^2 & \text{Square the first, twice the product, square the last. Our Solution} \end{array}$$

These two formulas will be important to commit to memory. The more familiar we are with them, the easier factoring, or multiplying in reverse, will be. The final example covers both types of problems (two perfect squares, one positive, one negative), be sure to notice the difference between the examples and how each formula is used

Example 255.

$$\begin{array}{ccc} (4x-7)(4x+7) & (4x+7)^2 & (4x-7)^2 \\ 16x^2 - 49 & 16x^2 + 56x + 49 & 16x^2 - 56x + 49 \end{array}$$

World View Note: There are also formulas for higher powers of binomials as well, such as $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. While French mathematician Blaise Pascal often gets credit for working with these expansions of binomials in the 17th century, Chinese mathematicians had been working with them almost 400 years earlier!

5.6 Practice - Multiply Special Products

Find each product.

1) $(x+8)(x-8)$

2) $(a-4)(a+4)$

3) $(1+3p)(1-3p)$

4) $(x-3)(x+3)$

5) $(1-7n)(1+7n)$

6) $(8m+5)(8m-5)$

7) $(5n-8)(5n+8)$

8) $(2r+3)(2r-3)$

9) $(4x+8)(4x-8)$

10) $(b-7)(b+7)$

11) $(4y-x)(4y+x)$

12) $(7a+7b)(7a-7b)$

13) $(4m-8n)(4m+8n)$

14) $(3y-3x)(3y+3x)$

15) $(6x-2y)(6x+2y)$

16) $(1+5n)^2$

17) $(a+5)^2$

18) $(v+4)^2$

19) $(x-8)^2$

20) $(1-6n)^2$

21) $(p+7)^2$

22) $(7k-7)^2$

23) $(7-5n)^2$

24) $(4x-5)^2$

25) $(5m-8)^2$

26) $(3a+3b)^2$

27) $(5x+7y)^2$

28) $(4m-n)^2$

29) $(2x+2y)^2$

30) $(8x+5y)^2$

31) $(5+2r)^2$

32) $(m-7)^2$

33) $(2+5x)^2$

34) $(8n+7)(8n-7)$

35) $(4v-7)\,(4v+7)$

36) $(b+4)(b-4)$

37) $(n-5)(n+5)$

38) $(7x+7)^2$

39) $(4k+2)^2$

40) $(3a-8)(3a+8)$

Chapter 6 : Factoring

6.1

Factoring - Greatest Common Factor

Objective: Find the greatest common factor of a polynomial and factor it out of the expression.

The opposite of multiplying polynomials together is factoring polynomials. There are many benifits of a polynomial being factored. We use factored polynomials to help us solve equations, learn behaviors of graphs, work with fractions and more. Because so many concepts in algebra depend on us being able to factor polynomials it is very important to have very strong factoring skills.

In this lesson we will focus on factoring using the greatest common factor or GCF of a polynomial. When we multiplied polynomials, we multiplied monomials by polynomials by distributing, solving problems such as $4x^2(2x^2 - 3x + 8) = 8x^4 - 12x^3 + 32x$. In this lesson we will work the same problem backwards. We will start with $8x^2 - 12x^3 + 32x$ and try and work backwards to the $4x^2(2x - 3x + 8)$.

To do this we have to be able to first identify what is the GCF of a polynomial. We will first introduce this by looking at finding the GCF of several numbers. To find a GCF of sevearal numbers we are looking for the largest number that can be divided by each of the numbers. This can often be done with quick mental math and it is shown in the following example

Example 262.

$$\begin{array}{rl}
\text{Find the GCF of } 15, 24, \text{and } 27 & \\
\dfrac{15}{3} = 5,\ \dfrac{24}{3} = 6,\ \dfrac{27}{3} = 9 & \text{Each of the numbers can be divided by } 3 \\
\text{GCF} = 3 & \text{Our Solution}
\end{array}$$

When there are variables in our problem we can first find the GCF of the numbers using mental math, then we take any variables that are in common with each term, using the lowest exponent. This is shown in the next example

Example 263.

$$\begin{array}{rl} \text{GCF of } 24x^4y^2z, 18x^2y^4, \text{and } 12x^3yz^5 & \\ \frac{24}{6}=4,\ \frac{18}{6}=3,\ \frac{12}{6}=2 & \text{Each number can be divided by 6} \\ x^2y & x \text{ and } y \text{ are in all 3, using lowest exponets} \\ \text{GCF} = 6x^2y & \text{Our Solution} \end{array}$$

To factor out a GCF from a polynomial we first need to identify the GCF of all the terms, this is the part that goes in front of the parenthesis, then we divide each term by the GCF, the answer is what is left inside the parenthesis. This is shown in the following examples

Example 264.

$$\begin{array}{rl} 4x^2-20x+16 & \text{GCF is 4, divide each term by 4} \\ \frac{4x^2}{4}=x^2,\ \frac{-20x}{4}=-5x,\ \frac{16}{4}=4 & \text{This is what is left inside the parenthesis} \\ 4(x^2-5x+4) & \text{Our Solution} \end{array}$$

With factoring we can always check our solutions by multiplying (distributing in this case) out the answer and the solution should be the original equation.

Example 265.

$$\begin{array}{rl} 25x^4-15x^3+20x^2 & \text{GCF is } 5x^2\text{, divide each term by this} \\ \frac{25x^4}{5x^2}=5x^2,\ \frac{-15x^3}{5x^2}=-3x,\ \frac{20x^2}{5x^2}=4 & \text{This is what is left inside the parenthesis} \\ 5x^2(5x^2-3x+4) & \text{Our Solution} \end{array}$$

Example 266.

$$\begin{array}{rl} 3x^3y^2z+5x^4y^3z^5-4xy^4 & \text{GCF is } xy^2\text{, divide each term by this} \end{array}$$

$$\frac{3x^3y^2z}{xy^2}=3x^2z, \frac{5x^4y^3z^5}{xy^2}=5x^3yz^5, \frac{-4xy^4}{xy^2}=-4y^2 \quad \text{This is what is left in parenthesis}$$
$$xy^2(3x^2z+5x^3yz^5-4y^2) \quad \text{Our Solution}$$

World View Note: The first recorded algorithm for finding the greatest common factor comes from Greek mathematician Euclid around the year 300 BC!

Example 267.

$$21x^3+14x^2+7x \quad \text{GCF is } 7x\text{, divide each term by this}$$
$$\frac{21x^3}{7x}=3x^2, \frac{14x^2}{7x}=2x, \frac{7x}{7x}=1 \quad \text{This is what is left inside the parenthesis}$$
$$7x(3x^2+2x+1) \quad \text{Our Solution}$$

It is important to note in the previous example, that when the GCF was $7x$ and $7x$ was one of the terms, dividing gave an answer of 1. Students often try to factor out the $7x$ and get zero which is incorrect, factoring will never make terms dissapear. Anything divided by itself is 1, be sure to not forget to put the 1 into the solution.

Often the second line is not shown in the work of factoring the GCF. We can simply identify the GCF and put it in front of the parenthesis as shown in the following two examples.

Example 268.

$$12x^5y^2-6x^4y^4+8x^3y^5 \quad \text{GCF is } 2x^3y^2\text{, put this in front of parenthesis and divide}$$
$$2x^3y^2(6x^2-3xy^2+4y^3) \quad \text{Our Solution}$$

Example 269.

$$18a^4b^3-27a^3b^3+9a^2b^3 \quad \text{GCF is } 9a^2b^3\text{, divide each term by this}$$
$$9a^2b^3(2a^2-3a+1) \quad \text{Our Solution}$$

Again, in the previous problem, when dividing $9a^2b^3$ by itself, the answer is 1, not zero. Be very careful that each term is accounted for in your final solution.

6.1 Practice - Greatest Common Factor

Factor the common factor out of each expression.

1) $9+8b^2$

2) $x-5$

3) $45x^2-25$

4) $1+2n^2$

5) $56-35p$

6) $50x-80y$

7) $7\text{ab}-35a^2b$

8) $27x^2y^5-72x^3y^2$

9) $-3a^2b+6a^3b^2$

10) $8x^3y^2+4x^3$

11) $-5x^2-5x^3-15x^4$

12) $-32n^9+32n^6+40n^5$

13) $20x^4-30x+30$

14) $21p^6+30p^2+27$

15) $28m^4+40m^3+8$

16) $-10x^4+20x^2+12x$

17) $30b^9+5ab-15a^2$

18) $27y^7+12y^2x+9y^2$

19) $-48a^2b^2-56a^3b-56a^5b$

20) $30m^6+15mn^2-25$

21) $20x^8y^2z^2+15x^5y^2z+35x^3y^3z$

22) $3p+12q-15q^2r^2$

23) $50x^2y+10y^2+70xz^2$

24) $30y^4z^3x^5+50y^4z^5-10y^4z^3x$

25) $30qpr-5qp+5q$

26) $28b+14b^2+35b^3+7b^5$

27) $-18n^5+3n^3-21n+3$

28) $30a^8+6a^5+27a^3+21a^2$

29) $-40x^{11}-20x^{12}+50x^{13}-50x^{14}$

30) $-24x^6-4x^4+12x^3+4x^2$

31) $-32mn^8+4m^6n+12mn^4+16mn$

32) $-10y^7+6y^{10}-4y^{10}x-8y^8x$

6.2

Factoring - Grouping

Objective: Factor polynomials with four terms using grouping.

The first thing we will always do when factoring is try to factor out a GCF. This GCF is often a monomial like in the problem $5xy + 10xz$ the GCF is the monomial $5x$, so we would have $5x(y + 2z)$. However, a GCF does not have to be a monomial, it could be a binomial. To see this, consider the following two example.

Example 270.

$$\begin{array}{rl} 3ax - 7bx & \text{Both have } x \text{ in common, factor it out} \\ x(3a - 7b) & \text{Our Solution} \end{array}$$

Now the same problem, but instead of x we have $(2a + 5b)$.

Example 271.

$$\begin{array}{rl} 3a(2a+5b) - 7b(2a+5b) & \text{Both have } (2a+5b) \text{ in common, factor it out} \\ (2a+5b)(3a-7b) & \text{Our Solution} \end{array}$$

In the same way we factored out a GCF of x we can factor out a GCF which is a binomial, $(2a + 5b)$. This process can be extended to factor problems where there is no GCF to factor out, or after the GCF is factored out, there is more factoring that can be done. Here we will have to use another strategy to factor. We will use a process known as grouping. Grouping is how we will factor if there are four terms in the problem. Remember, factoring is like multiplying in reverse, so first we will look at a multiplication problem and then try to reverse the process.

Example 272.

$$\begin{array}{rl} (2a+3)(5b+2) & \text{Distribute } (2a+3) \text{ into second parenthesis} \\ 5b(2a+3) + 2(2a+3) & \text{Distribute each monomial} \\ 10\text{ab} + 15b + 4a + 6 & \text{Our Solution} \end{array}$$

The solution has four terms in it. We arrived at the solution by looking at the two parts, $5b(2a + 3)$ and $2(2a + 3)$. When we are factoring by grouping we will always divide the problem into two parts, the first two terms and the last two terms. Then we can factor the GCF out of both the left and right sides. When we do this our hope is what is left in the parenthesis will match on both the left and right. If they match we can pull this matching GCF out front, putting the rest in parenthesis and we will be factored. The next example is the same problem worked backwards, factoring instead of multiplying.

Example 273.

$10\text{ab}+15b+4a+6$	Split problem into two groups
$\boxed{10\text{ab}+15b}\boxed{+4a+6}$	GCF on left is $5b$, on the right is 2
$\boxed{5b(2a+3)}\boxed{+2(2a+3)}$	$(2a+3)$ is the same! Factor out this GCF
$(2a+3)(5b+2)$	Our Solution

The key for grouping to work is after the GCF is factored out of the left and right, the two binomials must match exactly. If there is any difference between the two we either have to do some adjusting or it can't be factored using the grouping method. Consider the following example.

Example 274.

$6x^2+9xy-14x-21y$	Split problem into two groups
$\boxed{6x^2+9xy}\boxed{-14x-21y}$	GCF on left is $3x$, on right is 7
$\boxed{3x(2x+3y)}\boxed{+7(-2x-3y)}$	The signs in the parenthesis don't match!

when the signs don't match on both terms we can easily make them match by factoring the opposite of the GCF on the right side. Instead of 7 we will use -7. This will change the signs inside the second parenthesis.

$\boxed{3x(2x+3y)}\boxed{-7(2x+3y)}$	$(2x+3y)$ is the same! Factor out this GCF
$(2x+3y)(3x-7)$	Our Solution

Often we can recognize early that we need to use the opposite of the GCF when factoring. If the first term of the first binomial is positive in the problem, we will also want the first term of the second binomial to be positive. If it is negative then we will use the opposite of the GCF to be sure they match.

Example 275.

$5xy-8x-10y+16$	Split the problem into two groups
$\boxed{5xy-8x}\boxed{-10y+16}$	GCF on left is x, on right we need a negative,
	so we use -2
$\boxed{x(5y-8)}\boxed{-2(5y-8)}$	$(5y-8)$ is the same! Factor out this GCF
$(5y-8)(x-2)$	Our Solution

Sometimes when factoring the GCF out of the left or right side there is no GCF to factor out. In this case we will use either the GCF of 1 or -1. Often this is all we need to be sure the two binomials match.

Example 276.

$$\begin{array}{rl} 12ab-14a-6b+7 & \text{Split the problem into two groups} \\ \boxed{12\text{ab}-14a}\boxed{-6b+7} & \text{GCF on left is } 2a, \text{ on right, no GCF, use } -1 \\ \boxed{2a(6b-7)}\boxed{-1(6b-7)} & (6b-7) \text{ is the same! Factor out this GCF} \\ (6b-7)(2a-1) & \text{Our Solution} \end{array}$$

Example 277.

$$\begin{array}{rl} 6x^3-15x^2+2x-5 & \text{Split problem into two groups} \\ \boxed{6x^3-15x^2}\boxed{+2x-5} & \text{GCF on left is } 3x^2, \text{ on right, no GCF, use } 1 \\ \boxed{3x^2(2x-5)}\boxed{+1(2x-5)} & (2x-5) \text{ is the same! Factor out this GCF} \\ (2x-5)(3x^2+1) & \text{Our Solution} \end{array}$$

Another problem that may come up with grouping is after factoring out the GCF on the left and right, the binomials don't match, more than just the signs are different. In this case we may have to adjust the problem slightly. One way to do this is to change the order of the terms and try again. To do this we will move the second term to the end of the problem and see if that helps us use grouping.

Example 278.

$$\begin{array}{rl} 4a^2-21b^3+6ab-14ab^2 & \text{Split the problem into two groups} \\ \boxed{4a^2-21b^3}\boxed{+6ab-14ab^2} & \text{GCF on left is } 1, \text{ on right is } 2ab \\ \boxed{1(4a^2-21b^3)}\boxed{+2ab(3-7b)} & \text{Binomials don't match! Move second term to end} \\ 4a^2+6ab-14ab^2-21b^3 & \text{Start over, split the problem into two groups} \\ \boxed{4a^2+6\text{ab}}\boxed{-14ab^2-21b^3} & \text{GCF on left is } 2a, \text{ on right is } -7b^2 \\ \boxed{2a(2a+3b)}\boxed{-7b^2(2a+3b)} & (2a+3b) \text{ is the same! Factor out this GCF} \\ (2a+3b)(2a-7b^2) & \text{Our Solution} \end{array}$$

When rearranging terms the problem can still be out of order. Sometimes after factoring out the GCF the terms are backwards. There are two ways that this can happen, one with addition, one with subtraction. If it happens with addition, for example the binomials are $(a + b)$ and $(b + a)$, we don't have to do any extra work. This is because addition is the same in either order $(5+3=3+5=8)$.

Example 279.

$$\begin{array}{rl}
7+y-3xy-21x & \text{Split the problem into two groups}\\
\boxed{7+y}\boxed{-3xy-21x} & \text{GCF on left is } 1\text{, on the right is } -3x\\
\boxed{1(7+y)}\boxed{-3x(y+7)} & y+7 \text{ and } 7+y \text{ are the same, use either one}\\
(y+7)(1-3x) & \text{Our Solution}
\end{array}$$

However, if the binomial has subtraction, then we need to be a bit more careful. For example, if the binomials are $(a-b)$ and $(b-a)$, we will factor out the opposite of the GCF on one part, usually the second. Notice what happens when we factor out -1.

Example 280.

$$\begin{array}{rl}
(b-a) & \text{Factor out } -1\\
-1(-b+a) & \text{Addition can be in either order, switch order}\\
-1(a-b) & \text{The order of the subtraction has been switched!}
\end{array}$$

Generally we won't show all the above steps, we will simply factor out the opposite of the GCF and switch the order of the subtraction to make it match the other binomial.

Example 281.

$$\begin{array}{rl}
8xy-12y+15-10x & \text{Split the problem into two groups}\\
\boxed{8xy-12y}\boxed{15-10x} & \text{GCF on left is } 4y\text{, on right, } 5\\
\boxed{4y(2x-3)}\boxed{+5(3-2x)} & \text{Need to switch subtraction order, use } -5 \text{ in middle}\\
\boxed{4y(2y-3)}\boxed{-5(2x-3)} & \text{Now } 2x-3 \text{ match on both! Factor out this GCF}\\
(2x-3)(4y-5) & \text{Our Solution}
\end{array}$$

World View Note: Sofia Kovalevskaya of Russia was the first woman on the editorial staff of a mathematical journal in the late 19th century. She also did research on how the rings of Saturn rotated.

6.2 Practice - Grouping

Factor each completely.

1) $40r^3-8r^2-25r+5$

2) $35x^3-10x^2-56x+16$

3) $3n^3-2n^2-9n+6$

4) $14v^3+10v^2-7v-5$

5) $15b^3+21b^2-35b-49$

6) $6x^3-48x^2+5x-40$

7) $3x^3+15x^2+2x+10$

8) $28p^3+21p^2+20p+15$

9) $35x^3-28x^2-20x+16$

10) $7n^3+21n^2-5n-15$

11) $7xy-49x+5y-35$

12) $42r^3-49r^2+18r-21$

13) $32xy+40x^2+12y+15x$

14) $15ab-6a+5b^3-2b^2$

15) $16xy-56x+2y-7$

16) $3mn-8m+15n-40$

17) $2xy-8x^2+7y^3-28y^2x$

18) $5mn+2m-25n-10$

19) $40xy+35x-8y^2-7y$

20) $8xy+56x-y-7$

21) $32uv-20u+24v-15$

22) $4uv+14u^2+12v+42u$

23) $10xy+30+25x+12y$

24) $24xy+25y^2-20x-30y^3$

25) $3uv+14u-6u^2-7v$

26) $56ab+14-49a-16b$

27) $16xy-3x-6x^2+8y$

6.3

Factoring - Trinomials where a = 1

Objective: Factor trinomials where the coefficient of x^2 is one.

Factoring with three terms, or trinomials, is the most important type of factoring to be able to master. As factoring is multiplication backwards we will start with a multipication problem and look at how we can reverse the process.

Example 282.

$(x+6)(x-4)$	Distribute $(x+6)$ through second parenthesis
$x(x+6)-4(x+6)$	Distribute each monomial through parenthesis
$x^2+6x-4x-24$	Combine like terms
$x^2+2x-24$	Our Solution

You may notice that if you reverse the last three steps the process looks like grouping. This is because it is grouping! The GCF of the left two terms is x and the GCF of the second two terms is -4. The way we will factor trinomials is to make them into a polynomial with four terms and then factor by grouping. This is shown in the following example, the same problem worked backwards

Example 283.

$x^2+2x-24$	Split middle term into $+6x-4x$
$x^2+6x-4x-24$	Grouping: GCF on left is x, on right is -4
$x(x+6)-4(x+6)$	$(x+6)$ is the same, factor out this GCF
$(x+6)(x-4)$	Our Solution

The trick to make these problems work is how we split the middle term. Why did we pick $+6x-4x$ and not $+5x-3x$? The reason is because $6x-4x$ is the only combination that works! So how do we know what is the one combination that works? To find the correct way to split the middle term we will use what is called the ac method. In the next lesson we will discuss why it is called the ac method. The way the ac method works is we find a pair of numers that multiply to a certain number and add to another number. Here we will try to multiply to get the last term and add to get the coefficient of the middle term. In the previous example that would mean we wanted to multiply to -24 and add to $+2$. The only numbers that can do this are 6 and -4 ($6\cdot-4=-24$ and $6+(-4)=2$). This process is shown in the next few examples

Example 284.

$$\begin{array}{rl} x^2+9x+18 & \text{Want to multiply to } 18, \text{ add to } 9\\ x^2+6x+3x+18 & 6 \text{ and } 3, \text{ split the middle term}\\ x(x+6)+3(x+6) & \text{Factor by grouping}\\ (x+6)(x+3) & \text{Our Solution} \end{array}$$

Example 285.

$$\begin{array}{rl} x^2-4x+3 & \text{Want to multiply to } 3, \text{ add to } -4\\ x^2-3x-x+3 & -3 \text{ and } -1, \text{ split the middle term}\\ x(x-3)-1(x-3) & \text{Factor by grouping}\\ (x-3)(x-1) & \text{Our Solution} \end{array}$$

Example 286.

$$\begin{array}{rl} x^2-8x-20 & \text{Want to multiply to } -20, \text{ add to } -8\\ x^2-10x+2x-20 & -10 \text{ and } 2, \text{ split the middle term}\\ x(x-10)+2(x-10) & \text{Factor by grouping}\\ (x-10)(x+2) & \text{Our Solution} \end{array}$$

Often when factoring we have two variables. These problems solve just like problems with one variable, using the coefficients to decide how to split the middle term

Example 287.

$$\begin{array}{rl} a^2-9ab+14b^2 & \text{Want to multiply to } 14, \text{ add to } -9\\ a^2-7ab-2ab+14b^2 & -7 \text{ and } -2, \text{ split the middle term}\\ a(a-7b)-2b(a-7b) & \text{Factor by grouping}\\ (a-7b)(a-2b) & \text{Our Solution} \end{array}$$

As the past few examples illustrate, it is very important to be aware of negatives as we find the pair of numbers we will use to split the middle term. Consier the following example, done incorrectly, ignoring negative signs

Warning 288.

$$\begin{array}{rl} x^2+5x-6 & \text{Want to multiply to } 6, \text{add } 5 \\ x^2+2x+3x-6 & 2 \text{ and } 3, \text{split the middle term} \\ x(x+2)+3(x-2) & \text{Factor by grouping} \\ ??? & \text{Binomials do not match!} \end{array}$$

Because we did not use the negative sign with the six to find our pair of numbers, the binomials did not match and grouping was not able to work at the end. Now the problem will be done correctly.

Example 289.

$$\begin{array}{rl} x^2+5x-6 & \text{Want to multiply to } -6, \text{add to } 5 \\ x^2+6x-x-6 & 6 \text{ and } -1, \text{split the middle term} \\ x(x+6)-1(x+6) & \text{Factor by grouping} \\ (x+6)(x-1) & \text{Our Solution} \end{array}$$

You may have noticed a shortcut for factoring these problems. Once we identify the two numbers that are used to split the middle term, these are the two numbers in our factors! In the previous example, the numbers used to split the middle term were 6 and -1, our factors turned out to be $(x+6)(x-1)$. This pattern does not always work, so be careful getting in the habit of using it. We can use it however, when we have no number (technically we have a 1) in front of x^2. In all the problems we have factored in this lesson there is no number in front of x^2. If this is the case then we can use this shortcut. This is shown in the next few examples.

Example 290.

$$\begin{array}{rl} x^2-7x-18 & \text{Want to multiply to } -18, \text{add to } -7 \\ & -9 \text{ and } 2, \text{write the factors} \\ (x-9)(x+2) & \text{Our Solution} \end{array}$$

Example 291.

$$
\begin{array}{rl}
m^2 - mn - 30n^2 & \text{Want to multiply to} - 30, \text{add to} - 1 \\
 & 5 \text{ and} - 6, \text{write the factors, don't forget second variable} \\
(m + 5n)(m - 6n) & \text{Our Solution}
\end{array}
$$

It is possible to have a problem that does not factor. If there is no combination of numbers that multiplies and adds to the correct numbers, then we say we cannot factor the polynomial, or we say the polynomial is prime. This is shown in the following example.

Example 292.

$$
\begin{array}{rl}
x^2 + 2x + 6 & \text{Want to multiply to } 6, \text{add to } 2 \\
1 \cdot 6 \text{ and } 2 \cdot 3 & \text{Only possibilities to multiply to six, none add to } 2 \\
\text{Prime, can't factor} & \text{Our Solution}
\end{array}
$$

When factoring it is important not to forget about the GCF. If all the terms in a problem have a common factor we will want to first factor out the GCF before we factor using any other method.

Example 293.

$$
\begin{array}{rl}
3x^2 - 24x + 45 & \text{GCF of all terms is } 3, \text{factor this out} \\
3(x^2 - 8x + 15) & \text{Want to multiply to } 15, \text{add to} - 8 \\
 & -5 \text{ and} - 3, \text{write the factors} \\
3(x - 5)(x - 3) & \text{Our Solution}
\end{array}
$$

Again it is important to comment on the shortcut of jumping right to the factors, this only works if there is no coefficient on x^2. In the next lesson we will look at how this process changes slightly when we have a number in front of x^2. Be careful not to use this shortcut on all factoring problems!

World View Note: The first person to use letters for unknown values was Francois Vieta in 1591 in France. He used vowels to represent variables we are solving for, just as codes used letters to represent an unknown message.

6.3 Practice - Trinomials where a = 1

Factor each completely.

1) $p^2+17p+72$

2) x^2+x-72

3) n^2-9n+8

4) x^2+x-30

5) $x^2-9x-10$

6) $x^2+13x+40$

7) $b^2+12b+32$

8) $b^2-17b+70$

9) $x^2+3x-70$

10) $x^2+3x-18$

11) $n^2-8n+15$

12) $a^2-6a-27$

13) $p^2+15p+54$

14) $p^2+7p-30$

15) $n^2-15n+56$

16) $m^2-15mn+50n^2$

17) $u^2-8uv+15v^2$

18) $m^2-3mn-40n^2$

19) $m^2+2mn-8n^2$

20) $x^2+10xy+16y^2$

21) $x^2-11xy+18y^2$

22) $u^2-9uv+14v^2$

23) $x^2+xy-12y^2$

24) $x^2+14xy+45y^2$

25) $x^2+4xy-12y^2$

26) $4x^2+52x+168$

27) $5a^2+60a+100$

28) $5n^2-45n+40$

29) $6a^2+24a-192$

30) $5v^2+20v-25$

31) $6x^2+18xy+12y^2$

32) $5m^2+30mn-90n^2$

33) $6x^2+96xy+378y^2$

34) $6m^2-36mn-162n^2$

6.4

Factoring - Trinomials where a $\neq$ 1

Objective: Factor trinomials using the ac method when the coefficient of x^2 is not one.

When factoring trinomials we used the ac method to split the middle term and then factor by grouping. The ac method gets it's name from the general trinomial equation, $ax^2 + bx + c$, where a, b, and c are the numbers in front of x^2, x and the constant at the end respectively.

World View Note: It was French philosopher Rene Descartes who first used letters from the beginning of the alphabet to represent values we know (a, b, c) and letters from the end to represent letters we don't know and are solving for (x, y, z).

The ac method is named ac because we multiply $a \cdot c$ to find out what we want to multiply to. In the previous lesson we always multiplied to just c because there was no number in front of x^2. This meant the number was 1 and we were multiplying to $1c$ or just c. Now we will have a number in front of x^2 so we will be looking for numbers that multiply to ac and add to b. Other than this, the process will be the same.

Example 294.

$$\begin{array}{rl} 3x^2 + 11x + 6 & \text{Multiply to } ac \text{ or } (3)(6) = 18, \text{ add to } 11 \\ 3x^2 + 9x + 2x + 6 & \text{The numbers are 9 and 2, split the middle term} \\ 3x(x+3) + 2(x+3) & \text{Factor by grouping} \\ (x+3)(3x+2) & \text{Our Solution} \end{array}$$

When $a = 1$, or no coefficient in front of x^2, we were able to use a shortcut, using the numbers that split the middle term in the factors. The previous example illustrates an important point, the shortcut does not work when $a \neq 1$. We must go through all the steps of grouping in order to factor the problem.

Example 295.

$$\begin{array}{rl} 8x^2 - 2x - 15 & \text{Multiply to } ac \text{ or } (8)(-15) = -120, \text{ add to } -2 \\ 8x^2 - 12x + 10x - 15 & \text{The numbers are } -12 \text{ and } 10, \text{ split the middle term} \\ 4x(2x-3) + 5(2x-3) & \text{Factor by grouping} \\ (2x-3)(4x+5) & \text{Our Solution} \end{array}$$

Example 296.

$$\begin{array}{rl} 10x^2-27x+5 & \text{Multiply to } ac \text{ or } (10)(5)=50, \text{ add to } -27 \\ 10x^2-25x-2x+5 & \text{The numbers are } -25 \text{ and } -2, \text{ split the middle term} \\ 5x(2x-5)-1(2x-5) & \text{Factor by grouping} \\ (2x-5)(5x-1) & \text{Our Solution} \end{array}$$

The same process works with two variables in the problem

Example 297.

$$\begin{array}{rl} 4x^2-xy-5y^2 & \text{Multiply to } ac \text{ or } (4)(-5)=-20, \text{ add to } -1 \\ 4x^2+4xy-5xy-5y^2 & \text{The numbers are } 4 \text{ and } -5, \text{ split the middle term} \\ 4x(x+y)-5y(x+y) & \text{Factor by grouping} \\ (x+y)(4x-5y) & \text{Our Solution} \end{array}$$

As always, when factoring we will first look for a GCF before using any other method, including the ac method. Factoring out the GCF first also has the added bonus of making the numbers smaller so the ac method becomes easier.

Example 298.

$$\begin{array}{rl} 18x^3+33x^2-30x & \text{GCF} = 3x, \text{ factor this out first} \\ 3x[6x^2+11x-10] & \text{Multiply to } ac \text{ or } (6)(-10)=-60, \text{ add to } 11 \\ 3x[6x^2+15x-4x-10] & \text{The numbers are } 15 \text{ and } -4, \text{ split the middle term} \\ 3x[3x(2x+5)-2(2x+5)] & \text{Factor by grouping} \\ 3x(2x+5)(3x-2) & \text{Our Solution} \end{array}$$

As was the case with trinomials when $a=1$, not all trinomials can be factored. If there is no combinations that multiply and add correctly then we can say the trinomial is prime and cannot be factored.

Example 299.

$$\begin{array}{rl} 3x^2+2x-7 & \text{Multiply to } ac \text{ or } (3)(-7)=-21, \text{ add to } 2 \\ -3(7) \text{ and } -7(3) & \text{Only two ways to multiply to } -21, \text{ it doesn't add to } 2 \\ \text{Prime, cannot be factored} & \text{Our Solution} \end{array}$$

6.4 Practice - Trinomials where a $\neq$ 1

Factor each completely.

1) $7x^2-48x+36$

2) $7n^2-44n+12$

3) $7b^2+15b+2$

4) $7v^2-24v-16$

5) $5a^2-13a-28$

6) $5n^2-4n-20$

7) $2x^2-5x+2$

8) $3r^2-4r-4$

9) $2x^2+19x+35$

10) $7x^2+29x-30$

11) $2b^2-b-3$

12) $5k^2-26k+24$

13) $5k^2+13k+6$

14) $3r^2+16r+21$

15) $3x^2-17x+20$

16) $3u^2+13uv-10v^2$

17) $3x^2+17xy+10y^2$

18) $7x^2-2xy-5y^2$

19) $5x^2+28xy-49y^2$

20) $5u^2+31uv-28v^2$

21) $6x^2-39x-21$

22) $10a^2-54a-36$

23) $21k^2-87k-90$

24) $21n^2+45n-54$

25) $14x^2-60x+16$

26) $4r^2+r-3$

27) $6x^2+29x+20$

28) $6p^2+11p-7$

29) $4k^2-17k+4$

30) $4r^2+3r-7$

31) $4x^2+9xy+2y^2$

32) $4m^2+6mn+6n^2$

33) $4m^2-9mn-9n^2$

34) $4x^2-6xy+30y^2$

35) $4x^2+13xy+3y^2$

36) $18u^2-3uv-36v^2$

37) $12x^2+62xy+70y^2$

38) $16x^2+60xy+36y^2$

39) $24x^2-52xy+8y^2$

40) $12x^2+50xy+28y^2$

6.5

Factoring - Factoring Special Products

Objective: Identify and factor special products including a difference of squares, perfect squares, and sum and difference of cubes.

When factoring there are a few special products that, if we can recognize them, can help us factor polynomials. The first is one we have seen before. When multiplying special products we found that a sum and a difference could multiply to a difference of squares. Here we will use this special product to help us factor

$$\textbf{Difference of Squares: } \boldsymbol{a^2 - b^2 = (a+b)(a-b)}$$

If we are subtracting two perfect squares then it will always factor to the sum and difference of the square roots.

Example 300.

$$\begin{array}{rl} x^2-16 & \text{Subtracting two perfect squares, the square roots are } x \text{ and } 4 \\ (x+4)(x-4) & \text{Our Solution} \end{array}$$

Example 301.

$$\begin{array}{rl} 9a^2-25b^2 & \text{Subtracting two perfect squares, the square roots are } 3a \text{ and } 5b \\ (3a+5b)(3a-5b) & \text{Our Solution} \end{array}$$

It is important to note, that a sum of squares will never factor. It is always prime. This can be seen if we try to use the ac method to factor $x^2 + 36$.

Example 302.

$$\begin{array}{rl} x^2+36 & \text{No } bx \text{ term, we use } 0x. \\ x^2+0x+36 & \text{Multiply to } 36\text{, add to } 0 \\ 1\cdot 36, 2\cdot 18, 3\cdot 12, 4\cdot 9, 6\cdot 6 & \text{No combinations that multiply to } 36 \text{ add to } 0 \\ \text{Prime, cannot factor} & \text{Our Solution} \end{array}$$

It turns out that a sum of squares is always prime.

Sum of Squares: $a^2 + b^2 = \text{Prime}$

A great example where we see a sum of squares comes from factoring a difference of 4th powers. Because the square root of a fourth power is a square ($\sqrt{a^4} = a^2$), we can factor a difference of fourth powers just like we factor a difference of squares, to a sum and difference of the square roots. This will give us two factors, one which will be a prime sum of squares, and a second which will be a difference of squares which we can factor again. This is shown in the following examples.

Example 303.

$a^4 - b^4$	Difference of squares with roots a^2 and b^2
$(a^2 + b^2)(a^2 - b^2)$	The first factor is prime, the second is a difference of squares!
$(a^2 + b^2)(a + b)(a - b)$	Our Solution

Example 304.

$x^4 - 16$	Difference of squares with roots x^2 and 4
$(x^2 + 4)(x^2 - 4)$	The first factor is prime, the second is a difference of squares!
$(x^2 + 4)(x + 2)(x - 2)$	Our Solution

Another factoring shortcut is the perfect square. We had a shortcut for multiplying a perfect square which can be reversed to help us factor a perfect square

Perfect Square: $a^2 + 2\text{ab} + b^2 = (a + b)^2$

A perfect square can be difficult to recognize at first glance, but if we use the ac method and get two of the same numbers we know we have a perfect square. Then we can just factor using the square roots of the first and last terms and the sign from the middle. This is shown in the following examples.

Example 305.

$x^2 - 6x + 9$	Multiply to 9, add to -6
	The numbers are -3 and -3, the same! Perfect square
$(x - 3)^2$	Use square roots from first and last terms and sign from the middle

Example 306.

$$\begin{array}{rl} 4x^2+20xy+25y^2 & \text{Multiply to 100, add to 20} \\ & \text{The numbers are 10 and 10, the same! Perfect square} \\ (2x+5y)^2 & \text{Use square roots from first and last terms and sign from the middle} \end{array}$$

World View Note: The first known record of work with polynomials comes from the Chinese around 200 BC. Problems would be written as "three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop sold for 29 dou. This would be the polynomial (trinomial) $3x+2y+z=29$.

Another factoring shortcut has cubes. With cubes we can either do a sum or a difference of cubes. Both sum and difference of cubes have very similar factoring formulas

$$\textbf{Sum of Cubes: } \boldsymbol{a^3+b^3=(a+b)(a^2-\mathrm{ab}+b^2)}$$

$$\textbf{Difference of Cubes: } \boldsymbol{a^3-b^3=(a-b)(a^2+\mathrm{ab}+b^2)}$$

Comparing the formulas you may notice that the only difference is the signs in between the terms. One way to keep these two formulas straight is to think of SOAP. S stands for Same sign as the problem. If we have a sum of cubes, we add first, a difference of cubes we subtract first. O stands for Opposite sign. If we have a sum, then subtraction is the second sign, a difference would have addition for the second sign. Finally, AP stands for Always Positive. Both formulas end with addition. The following examples show factoring with cubes.

Example 307.

$$\begin{array}{rl} m^3-27 & \text{We have cube roots } m \text{ and } 3 \\ (m\quad 3)(m^2\quad 3m\quad 9) & \text{Use formula, use SOAP to fill in signs} \\ (m-3)(m^2+3m+9) & \text{Our Solution} \end{array}$$

Example 308.

$$\begin{array}{rl} 125p^3+8r^3 & \text{We have cube roots } 5p \text{ and } 2r \\ (5p\quad 2r)(25p^2\quad 10r\quad 4r^2) & \text{Use formula, use SOAP to fill in signs} \\ (5p+2r)(25p^2-10r+4r^2) & \text{Our Solution} \end{array}$$

The previous example illustrates an important point. When we fill in the trinomial's first and last terms we square the cube roots $5p$ and $2r$. Often students forget to square the number in addition to the variable. Notice that when done correctly, both get cubed.

Often after factoring a sum or difference of cubes, students want to factor the second factor, the trinomial further. As a general rule, this factor will always be prime (unless there is a GCF which should have been factored out before using cubes rule).

The following table sumarizes all of the shortcuts that we can use to factor special products

Factoring Special Products

Difference of Squares	$a^2 - b^2 = (a + b)(a - b)$
Sum of Squares	$a^2 + b^2 = \text{Prime}$
Perfect Square	$a^2 + 2ab + b^2 = (a + b)^2$
Sum of Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

As always, when factoring special products it is important to check for a GCF first. Only after checking for a GCF should we be using the special products. This is shown in the following examples

Example 309.

$72x^2 - 2$	GCF is 2
$2(36x^2 - 1)$	Difference of Squares, square roots are $6x$ and 1
$2(6x + 1)(6x - 1)$	Our Solution

Example 310.

$48x^2y - 24xy + 3y$	GCF is $3y$
$3y(16x^2 - 8x + 1)$	Multiply to 16 add to 8
	The numbers are 4 and 4, the same! Perfect Square
$3y(4x - 1)^2$	Our Solution

Example 311.

$128a^4b^2 + 54ab^5$	GCF is $2ab^2$
$2ab^2(64a^3 + 27b^3)$	Sum of cubes! Cube roots are $4a$ and $3b$
$2\mathrm{ab}^2(4a + 3b)(16a^2 - 12ab + 9b^2)$	Our Solution

6.5 Practice - Factoring Special Products

Factor each completely.

1) r^2-16

2) x^2-9

3) v^2-25

4) x^2-1

5) p^2-4

6) $4v^2-1$

7) $9k^2-4$

8) $9a^2-1$

9) $3x^2-27$

10) $5n^2-20$

11) $16x^2-36$

12) $125x^2+45y^2$

13) $18a^2-50b^2$

14) $4m^2+64n^2$

15) a^2-2a+1

16) k^2+4k+4

17) x^2+6x+9

18) $n^2-8n+16$

19) x^2-6x+9

20) k^2-4k+4

21) $25p^2-10p+1$

22) x^2+2x+1

23) $25a^2+30ab+9b^2$

24) $x^2+8xy+16y^2$

25) $4a^2-20ab+25b^2$

26) $18m^2-24mn+8n^2$

27) $8x^2-24xy+18y^2$

28) $20x^2+20xy+5y^2$

29) $8-m^3$

30) x^3+64

31) x^3-64

32) x^3+8

33) $216-u^3$

34) $125x^3-216$

35) $125a^3-64$

36) $64x^3-27$

37) $64x^3+27y^3$

38) $32m^3-108n^3$

39) $54x^3+250y^3$

40) $375m^3+648n^3$

41) a^4-81

42) x^4-256

43) $16-z^4$

44) n^4-1

45) x^4-y^4

46) $16a^4-b^4$

47) m^4-81b^4

48) $81c^4-16d^4$

6.6

Factoring - Factoring Strategy

Objective: Idenfity and use the correct method to factor various polynomials.

With so many different tools used to factor, it is easy to get lost as to which tool to use when. Here we will attempt to organize all the different factoring types we have seen. A large part of deciding how to solve a problem is based on how many terms are in the problem. For all problem types we will always try to factor out the GCF first.

Factoring Strategy (GCF First!!!!!)

- **2 terms:** sum or difference of squares or cubes:

  $a^2 - b^2 = (a+b)(a-b)$

  $a^2 + b^2 = \text{Prime}$

  $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

  $a^3 - b^3 = (a-b)(a^2 + \text{ab} + b^2)$

- **3 terms:** ac method, watch for perfect square!

  $a^2 + 2ab + b^2 = (a+b)^2$

  Multiply to ac and add to b

- **4 terms:** grouping

We will use the above strategy to factor each of the following examples. Here the emphasis will be on which strategy to use rather than the steps used in that method.

Example 312.

$$\begin{array}{rl} 4x^2 + 56xy + 196y^2 & \text{GCF first, 4} \\ 4(x^2 + 14xy + 49y^2) & \text{Three terms, try ac method, multiply to 49, add to 14} \\ & \text{7 and 7, perfect square!} \end{array}$$

$$4(x+7y)^2 \quad \text{Our Solution}$$

Example 313.

$$\begin{array}{rl} 5x^2y+15xy-35x^2-105x & \text{GCF first, } 5x \\ 5x(xy+3y-7x-21) & \text{Four terms, try grouping} \\ 5x[y(x+3)-7(x+3)] & (x+3) \text{ match!} \\ 5x(x+3)(y-7) & \text{Our Solution} \end{array}$$

Example 314.

$$\begin{array}{rl} 100x^2-400 & \text{GCF first, } 100 \\ 100(x^2-4) & \text{Two terms, difference of squares} \\ 100(x+4)(x-4) & \text{Our Solution} \end{array}$$

Example 315.

$$\begin{array}{rl} 108x^3y^2-39x^2y^2+3xy^2 & \text{GCF first, } 3xy^2 \\ 3xy^2(36x^2-13x+1) & \text{Thee terms, ac method, multiply to } 36\text{, add to } -13 \\ 3xy^2(36x^2-9x-4x+1) & -9 \text{ and } -4\text{, split middle term} \\ 3xy^2[9x(4x-1)-1(4x-1)] & \text{Factor by grouping} \\ 3xy^2(4x-1)(9x-1) & \text{Our Solution} \end{array}$$

World View Note: Variables originated in ancient Greece where Aristotle would use a single capital letter to represent a number.

Example 316.

$$\begin{array}{rl} 5+625y^3 & \text{GCF first, } 5 \\ 5(1+125y^3) & \text{Two terms, sum of cubes} \\ 5(1+5y)(1-5y+25y^2) & \text{Our Solution} \end{array}$$

It is important to be comfortable and confident not just with using all the factoring methods, but decided on which method to use. This is why practice is very important!

6.6 Practice - Factoring Strategy

Factor each completely.

1) $24az - 18ah + 60yz - 45yh$

2) $2x^2 - 11x + 15$

3) $5u^2 - 9uv + 4v^2$

4) $16x^2 + 48xy + 36y^2$

5) $-2x^3 + 128y^3$

6) $20uv - 60u^3 - 5xv + 15xu^2$

7) $5n^3 + 7n^2 - 6n$

8) $2x^3 + 5x^2y + 3y^2x$

9) $54u^3 - 16$

10) $54 - 128x^3$

11) $n^2 - n$

12) $5x^2 - 22x - 15$

13) $x^2 - 4xy + 3y^2$

14) $45u^2 - 150uv + 125v^2$

15) $9x^2 - 25y^2$

16) $x^3 - 27y^3$

17) $m^2 - 4n^2$

18) $12ab - 18a + 6nb - 9n$

19) $36b^2c - 16xd - 24b^2d + 24xc$

20) $3m^3 - 6m^2n - 24n^2m$

21) $128 + 54x^3$

22) $64m^3 + 27n^3$

23) $2x^3 + 6x^2y - 20y^2x$

24) $3ac + 15ad^2 + x^2c + 5x^2d^2$

25) $n^3 + 7n^2 + 10n$

26) $64m^3 - n^3$

27) $27x^3 - 64$

28) $16a^2 - 9b^2$

29) $5x^2 + 2x$

30) $2x^2 - 10x + 12$

31) $3k^3 - 27k^2 + 60k$

32) $32x^2 - 18y^2$

33) $mn - 12x + 3m - 4xn$

34) $2k^2 + k - 10$

35) $16x^2 - 8\text{xy} + y^2$

36) $v^2 + v$

37) $27m^2 - 48n^2$

38) $x^3 + 4x^2$

39) $9x^3 + 21x^2y - 60y^2x$

40) $9n^3 - 3n^2$

41) $2m^2 + 6mn - 20n^2$

42) $2u^2v^2 - 11uv^3 + 15v^4$

6.7

Factoring - Solve by Factoring

Objective: Solve quadratic equation by factoring and using the zero product rule.

When solving linear equations such as $2x - 5 = 21$ we can solve for the variable directly by adding 5 and dividing by 2 to get 13. However, when we have x^2 (or a higher power of x) we cannot just isolate the variable as we did with the linear equations. One method that we can use to solve for the varaible is known as the zero product rule

Zero Product Rule: If $ab = 0$ then either $a = 0$ or $b = 0$

The zero product rule tells us that if two factors are multiplied together and the answer is zero, then one of the factors must be zero. We can use this to help us solve factored polynomials as in the following example.

Example 317.

$$\begin{array}{rl}
(2x-3)(5x+1)=0 & \text{One factor must be zero}\\
2x-3=0 \text{ or } 5x+1=0 & \text{Set each factor equal to zero}\\
\underline{+3+3} \qquad \underline{\quad -1-1} & \text{Solve each equation}\\
2x=3 \text{ or } 5x=-1 & \\
\overline{2}\ \ \overline{2} \qquad \overline{5}\ \ \overline{5} & \\
x=\dfrac{3}{2} \text{ or } \dfrac{-1}{5} & \text{Our Solution}
\end{array}$$

For the zero product rule to work we must have factors to set equal to zero. This means if the problem is not already factored we will factor it first.

Example 318.

$$\begin{array}{rl}
4x^2+x-3=0 & \text{Factor using the ac method, multiply to } -12\text{, add to } 1\\
4x^2-3x+4x-3=0 & \text{The numbers are } -3 \text{ and } 4\text{, split the middle term}\\
x(4x-3)+1(4x-3)=0 & \text{Factor by grouping}\\
(4x-3)(x+1)=0 & \text{One factor must be zero}\\
4x-3=0 \text{ or } x+1=0 & \text{Set each factor equal to zero}
\end{array}$$

$$\begin{array}{rl} \underline{+3+3} \quad \underline{-1-1} & \text{Solve each equation} \\ 4x=3 \text{ or } x=-1 & \\ \overline{4}\ \overline{4} & \\ x=\frac{3}{4} \text{ or } -1 & \text{Our Solution} \end{array}$$

Another important part of the zero product rule is that before we factor, the equation must equal zero. If it does not, we must move terms around so it does equal zero. Generally we like the x^2 term to be positive.

Example 319.

$$\begin{array}{rl} x^2=8x-15 & \text{Set equal to zero by moving terms to the left} \\ \underline{-8x+15} \quad \underline{-8x+15} & \\ x^2-8x+15=0 & \text{Factor using the ac method, multiply to 15, add to} -8 \\ (x-5)(x-3)=0 & \text{The numbers are} -5 \text{ and} -3 \\ x-5=0 \text{ or } x-3=0 & \text{Set each factor equal to zero} \\ \underline{+5+5} \quad \underline{+3+3} & \text{Solve each equation} \\ x=5 \text{ or } x=3 & \text{Our Solution} \end{array}$$

Example 320.

$$\begin{array}{rl} (x-7)(x+3)=-9 & \text{Not equal to zero, multiply first, use FOIL} \\ x^2-7x+3x-21=-9 & \text{Combine like terms} \\ x^2-4x-21=-9 & \text{Move} -9 \text{ to other side so equation equals zero} \\ \underline{+9 \quad +9} & \\ x^2-4x-12=0 & \text{Factor using the ac method, mutiply to} -12\text{, add to} -4 \\ (x-6)(x+2)=0 & \text{The numbers are 6 and} -2 \\ x-6=0 \text{ or } x+2=0 & \text{Set each factor equal to zero} \\ \underline{+6+6} \quad \underline{-2-2} & \text{Solve each equation} \\ x=6 \text{ or } -2 & \text{Our Solution} \end{array}$$

Example 321.

$$\begin{array}{rl} 3x^2+4x-5=7x^2+4x-14 & \text{Set equal to zero by moving terms to the right} \\ \underline{-3x^2-4x+5} \quad \underline{-3x^2-4x+5} & \\ 0=4x^2-9 & \text{Factor using difference of squares} \end{array}$$

$$
\begin{array}{rl}
0=(2x+3)(2x-3) & \text{One factor must be zero} \\
2x+3=0 \ \text{ or } \ 2x-3=0 & \text{Set each factor equal to zero} \\
\underline{-3-3} \qquad\qquad \underline{+3+3} & \text{Solve each equation} \\
2x=-3 \ \text{ or } \ 2x=3 & \\
\overline{2} \quad \overline{2} \qquad \overline{2} \ \overline{2} & \\
x=\frac{-3}{2} \ \text{ or } \ \frac{3}{2} & \text{Our Solution}
\end{array}
$$

Most problems with x^2 will have two unique solutions. However, it is possible to have only one solution as the next example illustrates.

Example 322.

$$
\begin{array}{rl}
4x^2=12x-9 & \text{Set equal to zero by moving terms to left} \\
\underline{-12x+9} \quad \underline{-12x+9} & \\
4x^2-12x+9=0 & \text{Factor using the ac method, multiply to 36, add to} -12 \\
(2x-3)^2=0 & -6 \text{ and } -6, a \text{ perfect square!} \\
2x-3=0 & \text{Set this factor equal to zero} \\
\underline{+3+3} & \text{Solve the equation} \\
2x=3 & \\
\overline{2} \quad \overline{2} & \\
x=\frac{3}{2} & \text{Our Solution}
\end{array}
$$

As always it will be important to factor out the GCF first if we have one. This GCF is also a factor and must also be set equal to zero using the zero product rule. This may give us more than just two solution. The next few examples illustrate this.

Example 323.

$$
\begin{array}{rl}
4x^2=8x & \text{Set equal to zero by moving the terms to left} \\
-8x-8x & \text{Be careful, on the right side, they are not like terms!} \\
4x^2-8x=0 & \text{Factor out the GCF of } 4x \\
4x(x-2)=0 & \text{One factor must be zero} \\
4x=0 \ \text{ or } \ x-2=0 & \text{Set each factor equal to zero} \\
\overline{4} \quad \overline{4} \qquad \underline{+2+2} & \text{Solve each equation} \\
x=0 \ \text{ or } \ 2 & \text{Our Solution}
\end{array}
$$

Example 324.

$$
\begin{array}{rl}
2x^3-14x^2+24x=0 & \text{Factor out the GCF of } 2x\\
2x(x^2-7x+12)=0 & \text{Factor with ac method, multiply to } 12, \text{ add to } -7\\
2x(x-3)(x-4)=0 & \text{The numbers are } -3 \text{ and } -4\\
2x=0 \text{ or } x-3=0 \text{ or } x-4=0 & \text{Set each factor equal to zero}\\
\overline{2}\ \ \overline{2} \qquad \underline{+3+3} \qquad \underline{+4+4} & \text{Solve each equation}\\
x=0 \text{ or } 3 \text{ or } 4 & \text{Our Solutions}
\end{array}
$$

Example 325.

$$
\begin{array}{rl}
6x^2+21x-27=0 & \text{Factor out the GCF of } 3\\
3(2x^2+7x-9)=0 & \text{Factor with ac method, multiply to } -18, \text{ add to } 7\\
3(2x^2+9x-2x-9)=0 & \text{The numbers are } 9 \text{ and } -2\\
3[x(2x+9)-1(2x+9)]=0 & \text{Factor by grouping}\\
3(2x+9)(x-1)=0 & \text{One factor must be zero}\\
3=0 \text{ or } 2x+9=0 \text{ or } x-1=0 & \text{Set each factor equal to zero}\\
3\neq 0 \qquad \underline{-9-9} \qquad \underline{+1+1} & \text{Solve each equation}\\
2x=-9 \text{ or } x=1 & \\
\overline{2}\ \ \overline{2} & \\
x=-\dfrac{9}{2} \text{ or } 1 & \text{Our Solution}
\end{array}
$$

In the previous example, the GCF did not have a variable in it. When we set this factor equal to zero we got a false statement. No solutions come from this factor. Often a student will skip setting the GCF factor equal to zero if there is no variables in the GCF.

Just as not all polynomials cannot factor, all equations cannot be solved by factoring. If an equation does not factor we will have to solve it using another method. These other methods are saved for another section.

World View Note: While factoring works great to solve problems with x^2, Tartaglia, in 16th century Italy, developed a method to solve problems with x^3. He kept his method a secret until another mathematician, Cardan, talked him out of his secret and published the results. To this day the formula is known as Cardan's Formula.

A question often asked is if it is possible to get rid of the square on the variable by taking the square root of both sides. While it is possible, there are a few properties of square roots that we have not covered yet and thus it is common to break a rule of roots that we are not aware of at this point. The short reason we want to avoid this for now is because taking a square root will only give us one of the two answers. When we talk about roots we will come back to problems like these and see how we can solve using square roots in a method called completing the square. For now, **never** take the square root of both sides!

6.7 Practice - Solve by Factoring

Solve each equation by factoring.

1) $(k-7)(k+2)=0$

2) $(a+4)(a-3)=0$

3) $(x-1)(x+4)=0$

4) $(2x+5)(x-7)=0$

5) $6x^2-150=0$

6) $p^2+4p-32=0$

7) $2n^2+10n-28=0$

8) $m^2-m-30=0$

9) $7x^2+26x+15=0$

10) $40r^2-285r-280=0$

11) $5n^2-9n-2=0$

12) $2b^2-3b-2=0$

13) $x^2-4x-8=-8$

14) $v^2-8v-3=-3$

15) $x^2-5x-1=-5$

16) $a^2-6a+6=-2$

17) $49p^2+371p-163=5$

18) $7k^2+57k+13=5$

19) $7x^2+17x-20=-8$

20) $4n^2-13n+8=5$

21) $7r^2+84=-49r$

22) $7m^2-224=28m$

23) $x^2-6x=16$

24) $7n^2-28n=0$

25) $3v^2+7v=40$

26) $6b^2=5+7b$

27) $35x^2+120x=-45$

28) $9n^2+39n=-36$

29) $4k^2+18k-23=6k-7$

30) $a^2+7a-9=-3+6a$

31) $9x^2-46+7x=7x+8x^2+3$

32) $x^2+10x+30=6$

33) $2m^2+19m+40=-2m$

34) $5n^2+41n+40=-2$

35) $40p^2+183p-168=p+5p^2$

36) $24x^2+11x-80=3x$

Answers - Chapter 0

0.0

0.0 Sets of Numbers

1: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

3: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

5: {2, 4, 6, 8, 10}

7: Rational

9: Rational

11: Rational

13: Irrational

15: Irrational

17: Integer, Rational

19: Rational

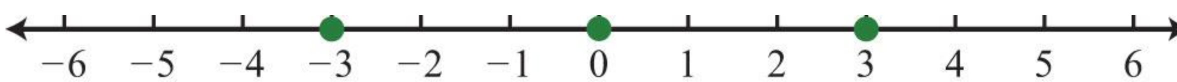
21: Integer, Rational

23: True

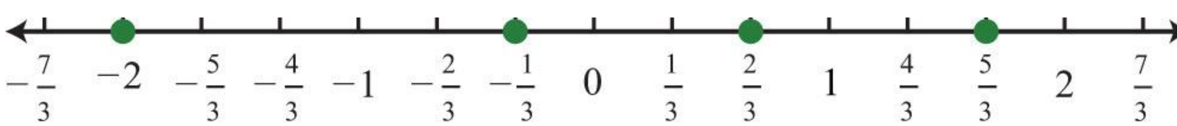
25: False

27: True

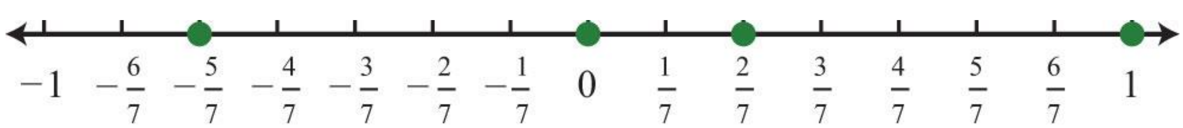
29.



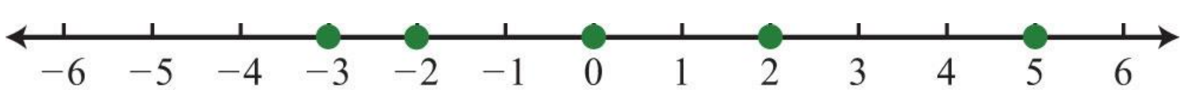
31.



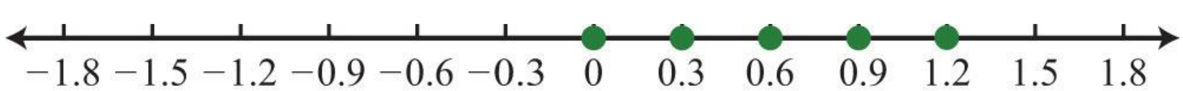
33.



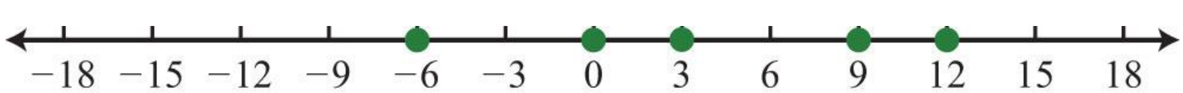
35.



37.



39.



41: <

43: >

45: >

47: <

49: =

51: True

53: False

55: True

57: True

59: True

61: −10, −7, −6 (answers may vary)

63: −1, −2/3, −1/3 (answers may vary)

65: −15, −10, −7 (answers may vary)

67: Ten is less than twenty.

69: Negative four is not equal to zero.

71: Zero is equal to zero.

73: $-7<0$

75: $0\geq-1$

77: $-2=-2$

79: 9

81: −10

83: −5

85: 1

87: 1

89: 11

91: $-\pi$

93: Larger

95: <

97: >

99: <

101: 20

103: 33

105: 2/5

107: 0

109: −12

111: −20

113: −7

115: 8/9

117: −3

119: 45

121: 6

123: −5

125: 9

127: ± 9

129: 0

131: Ø, No solution

133: ± 8

135: >

137: <

139: <

0.1

0.1 Answers - Integers

1) -2
2) 5
3) 2
4) 2
5) -6
6) -5
7) 8
8) 0
9) -2
10) -5
11) 4
12) -7
13) 3
14) -9
15) -2
16) -9
17) -1
18) -2
19) -3
20) 2
21) -7
22) 0
23) 11
24) 9
25) -3
26) -4
27) -3
28) 4
29) 0
30) -8
31) -4
32) -35
33) -80
34) 14
35) 8
36) 6
37) -56
38) -6
39) -36
40) 63
41) -10
42) 4
43) -20
44) 27
45) -24
46) -3
47) 7
48) 3
49) 2
50) 5
51) 2
52) 9
53) 7
54) -10
55) 4
56) 10
57) -8
58) 6
59) -6
60) -9

0.2

0.2 Answers - Fractions

1) $\frac{7}{2}$
2) $\frac{5}{4}$
3) $\frac{7}{5}$
4) $\frac{8}{3}$
5) $\frac{3}{2}$
6) $\frac{5}{4}$

7) $\frac{5}{4}$

8) $\frac{4}{3}$

9) $\frac{3}{2}$

10) $\frac{8}{3}$

11) $\frac{5}{2}$

12) $\frac{8}{7}$

13) $\frac{7}{2}$

14) $\frac{4}{3}$

15) $\frac{4}{3}$

16) $\frac{3}{2}$

17) $\frac{6}{5}$

18) $\frac{7}{6}$

19) $\frac{3}{2}$

20) $\frac{8}{7}$

21) 8

22) $\frac{5}{3}$

23) $-\frac{4}{9}$

24) $-\frac{2}{3}$

25) $-\frac{13}{4}$

26) $\frac{3}{4}$

27) $\frac{33}{20}$

28) $\frac{33}{56}$

29) 4

30) $\frac{18}{7}$

31) $\frac{1}{2}$

32) $-\frac{19}{20}$

33) 3

34) $-\frac{17}{15}$

35) $-\frac{7}{10}$

36) $\frac{5}{14}$

37) $-\frac{8}{7}$

38) $\frac{20}{21}$

39) $\frac{2}{9}$

40) $\frac{4}{3}$

41) $-\frac{21}{26}$

42) $\frac{25}{21}$

43) $-\frac{3}{2}$

44) $-\frac{5}{27}$

45) $\frac{40}{9}$

46) $-\frac{1}{10}$

47) $-\frac{45}{7}$

48) $\frac{13}{15}$

49) $\frac{4}{27}$

50) $\frac{32}{65}$

51) $\frac{1}{15}$

52) 1

53) -1

54) $-\frac{10}{7}$

55) $\frac{2}{7}$

56) 2

57) 3

58) $-\frac{31}{8}$

59) $\frac{37}{20}$

60) $-\frac{5}{3}$

61) $\frac{33}{20}$

62) $\frac{3}{7}$

63) $\frac{47}{56}$

64) $-\frac{7}{6}$

65) $\frac{2}{3}$

66) $-\frac{4}{3}$

67) 1

68) $\frac{7}{8}$

69) $\frac{19}{20}$

70) $-\frac{2}{5}$

71) $-\frac{145}{56}$

72) $-\frac{29}{15}$

73) $\frac{34}{7}$

74) $-\frac{23}{3}$

75) $-\frac{3}{8}$

76) $-\frac{2}{3}$

77) $-\frac{5}{24}$

78) $\frac{39}{14}$

79) $-\frac{5}{6}$

80) $\frac{1}{10}$

81) 2

82) $\frac{62}{21}$

0.3

0.3 Answers - Order of Operation

1) 24
2) -1
3) 5
4) 180
5) 4
6) 8
7) 1
8) 8
9) 6
10) -6
11) -10
12) -9
13) 20
14) -22
15) 2
16) 28
17) -40
18) -15
19) 3
20) 0
21) -18
22) -3
23) -4
24) 3
25) 2

0.4

0.4 Answers - Properties of Algebra

1) 7
2) 29
3) 1
4) 3
5) 23
6) 14
7) 25
8) 46
9) 7
10) 8
11) 5
12) 10
13) 1
14) 6
15) 1
16) 2
17) 36
18) 54
19) 7
20) 38
21) $r+1$
22) $-4x-2$
23) $2n$
24) $11b+7$
25) $15v$
26) $7x$
27) $-9x$
28) $-7a-1$
29) $k+5$
30) $-3p$
31) $-5x-9$
32) $-9-10n$
33) $-m$
34) $-5-r$
35) $10n+3$
36) $5b$
37) $-8x+32$
38) $24v+27$
39) $8n^2+72n$
40) $5-9a$
41) $-7k^2+42k$
42) $10x+20x^2$
43) $-6-36x$
44) $-2n-2$
45) $40m-8m^2$
46) $-18p^2+2p$
47) $-36x+9x^2$
48) $32n-8$
49) $-9b^2+90b$
50) $-4-28r$
51) $-40n-80n^2$
52) $16x^2-20x$
53) $14b+90$

54) $60v - 7$

55) $-3x + 8x^2$

56) $-89x + 81$

57) $-68k^2 - 8k$

58) $-19 - 90a$

59) $-34 - 49p$

60) $-10x + 17$

61) $10 - 4n$

62) $-30 + 9m$

63) $12x + 60$

64) $30r - 16r^2$

65) $-72n - 48 - 8n^2$

66) $-42b - 45 - 4b^2$

67) $79 - 79v$

68) $-8x + 22$

69) $-20n^2 + 80n - 42$

70) $-12 + 57a + 54a^2$

71) $-75 - 20k$

72) $-128x - 121$

73) $4n^2 - 3n - 5$

74) $2x^2 - 6x - 3$

75) $4p - 5$

76) $3x^2 + 7x - 7$

77) $-v^2 + 2v + 2$

78) $-7b^2 + 3b - 8$

79) $-4k^2 + 12$

80) $a^2 + 3a$

81) $3x^2 - 15$

82) $-n^2 + 6$

Answers - Chapter 1

1.1

1.1 Answers to One-Step Equations

1) 7

2) 11

3) -5

4) 4

5) 10

6) 6

7) -19

8) -6

9) 18

10) 6

11) -20

12) -7

13) -108

14) 5

15) -8

16) 4

17) 17

18) 4

19) 20

20) -208

21) 3

22) 16

23) -13

24) -9

25) 15

26) 8

27) -10

28) -204

29) 5

30) 2

31) -11

32) -14

33) 14

34) 1

35) -11

36) -15

37) -240

38) -135

39) -16

40) -380

1.2

1.2 Answers to Two-Step Equations

1) -4

2) 7

3) -14
4) -2
5) 10
6) -12
7) 0
8) 12
9) -10
10) -16
11) 14
12) -7
13) 4
14) -5
15) 16
16) -15
17) 7
18) 12
19) 9
20) 0
21) 11
22) -6
23) -10
24) 13
25) 1
26) 4
27) -9
28) 15
29) -6
30) 6
31) -16
32) -4
33) 8
34) -13
35) -2
36) 10
37) -12
38) 0
39) 12
40) -9

1.3

1.3 Answers to General Linear Equations

1) -3
2) 6
3) 7
4) 0
5) 1
6) 3
7) 5
8) -4
9) 0
10) 3
11) 1
12) All real numbers
13) 8
14) 1
15) -7
16) 0
17) 2
18) -3
19) -3
20) 3
21) 3
22) -1
23) -1
24) -1
25) 8
26) 0
27) -1
28) 5
29) -1
30) 1
31) -4
32) 0
33) -3
34) 0
35) 0
36) -2
37) -6
38) -3
39) 5
40) 6
41) 0
42) -2
43) No Solution
44) 0

45) 12

46) All real numbers

47) No Solution

48) 1

49) -9

50) 0

1.4

1.4 Answers to Solving with Fractions

1) $\frac{3}{4}$

2) $-\frac{4}{3}$

3) $\frac{6}{5}$

4) $\frac{1}{6}$

5) $-\frac{19}{6}$

6) $\frac{25}{8}$

7) $-\frac{7}{9}$

8) $-\frac{1}{3}$

9) -2

10) $\frac{3}{2}$

11) 0

12) $\frac{4}{3}$

13) $-\frac{3}{2}$

14) $\frac{1}{2}$

15) $-\frac{4}{3}$

16) 1

17) 0

18) $-\frac{5}{3}$

19) 1

20) 1

21) $\frac{1}{2}$

22) -1

23) -2

24) $-\frac{9}{4}$

25) 16

26) $-\frac{1}{2}$

27) $-\frac{5}{3}$

28) $-\frac{3}{2}$

29) $\frac{4}{3}$

30) $\frac{3}{2}$

1.5

1.5 Answers - Formulas

1. $b=\frac{c}{a}$
2. h = gi
3. $x=\frac{\text{gb}}{f}$
4. $y=\frac{\text{pq}}{3}$
5. $x=\frac{a}{3b}$
6. $y=\frac{\text{cb}}{\text{dm}}$
7. $m=\frac{E}{c^2}$
8. $D=\frac{\text{ds}}{S}$
9. $\pi=\frac{3V}{4r^3}$
10. $m=\frac{2E}{v_2}$
11. $c=b-a$
12. x = g + f
13. $y=\frac{\text{cm}+\text{cn}}{4}$
14. $r=\frac{k(a-3)}{5}$
15. $D=\frac{12V}{\pi n}$
16. $k=\frac{F}{R-L}$
17. $n=\frac{P}{p-c}$
18. $L=S-2B$
19. $D=\text{TL}+d$
20. $E_a=\text{IR}+\text{Eg}$
21. $L_o=\frac{L}{1+\text{at}}$
22. $x=\frac{c-b}{a}$
23. $m=\frac{p-q}{2}$
24. $L=\frac{q+6p}{6}$
25. k = qr + m
26. $T=\frac{R-b}{a}$
27. $v=\frac{16t^2+h}{t}$
28. $h=\frac{s-\pi r^2}{\pi r}$
29. $Q_2=\frac{Q_1+\text{PQ}_1}{P}$

30. $r_1 = \frac{L - 2d - \pi r^2}{\pi}$
31. $T_1 = \frac{\mathrm{Rd} - \mathrm{kAT}_2}{\mathrm{kA}}$
32. $v_2 = \frac{\mathrm{Pg} + V_1^2}{V_1}$
33. $a = \frac{c - b}{x}$
34. $r = \frac{d}{t}$
35. $w = \frac{V}{\ell h}$
36. $h = \frac{3v}{\pi r^2}$
37. $a = \frac{c - 1}{b}$
38. $b = \frac{c - 1}{a}$
39. $t = \frac{5 + \mathrm{bw}}{a}$
40. $w = \frac{\mathrm{at} - s}{b}$
41. $x = \frac{c - \mathrm{bx}}{x}$
42. $x = 3 - 5y$
43. $y = \frac{3 - x}{5}$
44. $x = \frac{7 - 2y}{3}$
45. $y = \frac{7 - 3x}{2}$
46. $a = \frac{7b + 4}{5}$
47. $b = \frac{5a - 4}{7}$
48. $x = \frac{8 + 5y}{4}$
49. $y = \frac{4x - 8}{5}$
50. $f = \frac{9c + 160}{5}$

1.6

1.6 Answers to Absolute Value Equations

1) $8, \ -8$
2) $7, -7$
3) $1, -1$
4) $2, \ -2$
5) $6, \ -\frac{29}{4}$
6) $\frac{38}{9}, -6$
7) $-2, -\frac{10}{3}$
8) $-3, 9$
9) $3, \ -\frac{39}{7}$
10) $\frac{16}{5}, -6$
11) $7, \ -\frac{29}{3}$
12) $-\frac{1}{3}, -1$
13) $-9, 15$
14) $3, -\frac{5}{3}$
15) $-2, 0$
16) $0, \ -2$
17) $-\frac{6}{7}, 0$
18) $-4, \frac{4}{3}$
19) $-\frac{17}{2}, \frac{7}{2}$
20) $-\frac{6}{5}, -2$
21) $-6, -8$
22) $6, -\frac{25}{3}$
23) $1, \ -\frac{13}{7}$
24) $7, -21$
25) $-2, 10$
26) $-\frac{7}{5}, 1$
27) $6, \ -\frac{16}{3}$
28) $\frac{2}{5}, 0$
29) $-\frac{13}{7}, 1$
30) $-3, 5$
31) $-\frac{4}{3}, -\frac{2}{7}$
32) $-6, \frac{2}{5}$
33) $7, \frac{1}{5}$
34) $-\frac{22}{5}, -\frac{2}{13}$
35) $-\frac{19}{22}, -\frac{11}{38}$
36) $0, -\frac{12}{5}$

1.7

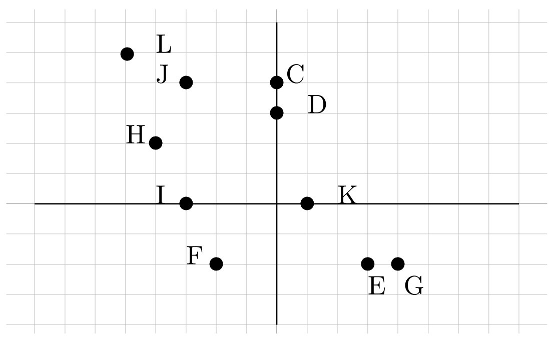
2.1 (Part 1)

Answers - Chapter 2

2.1 (Part 1) Answers - Points and Lines

1) B(4, -3) C(1, 2) D($-1, 4$)
E($-5, 0$) F(2, -3) G(1, 3)
H($-1, -4$) I($-2, -1$) J(0, 2)
K($-4, 3$)

2)



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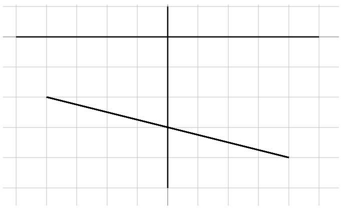

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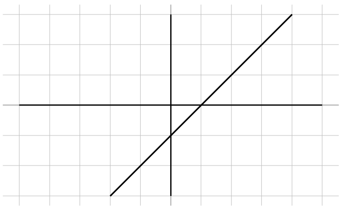

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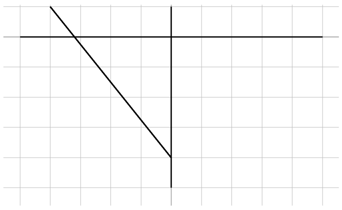

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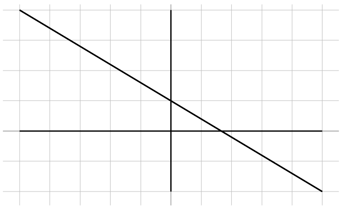

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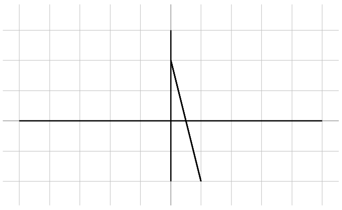

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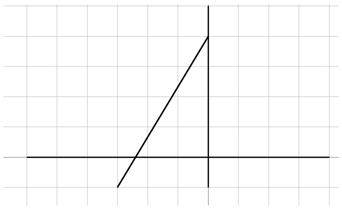

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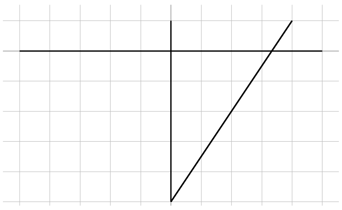

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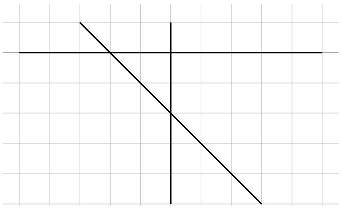

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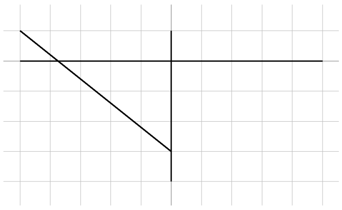

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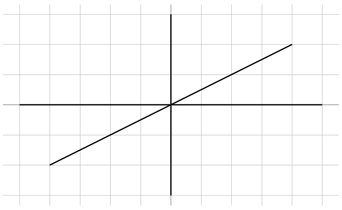

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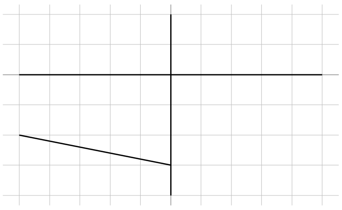

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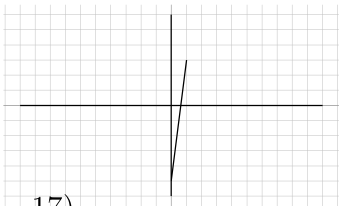

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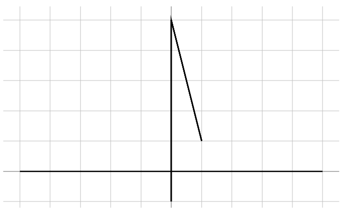

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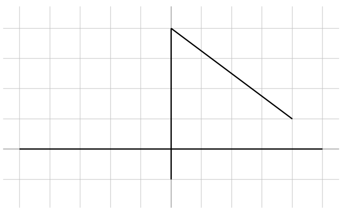

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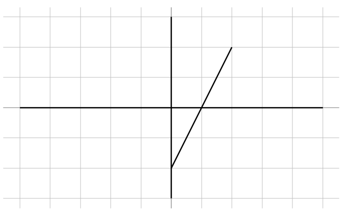

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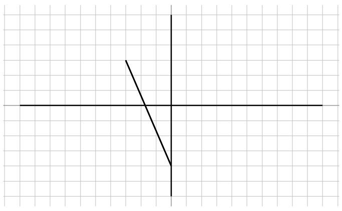

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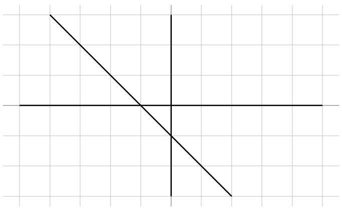

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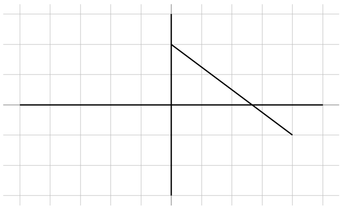

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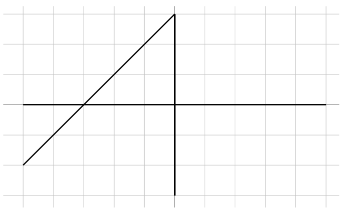

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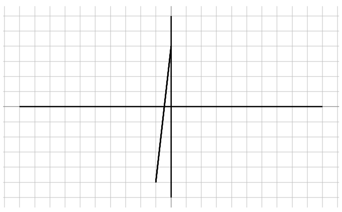
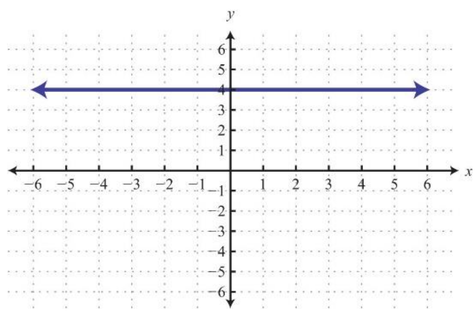


2.1 (Part 2) Answers – Horizontal and Vertical Lines

1: Yes

3: No

5:



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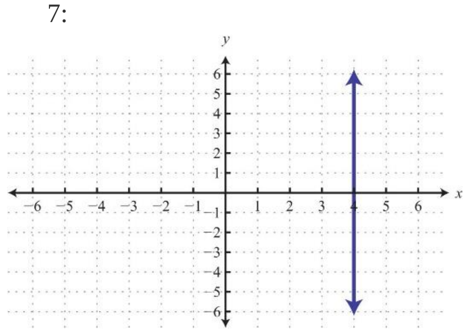

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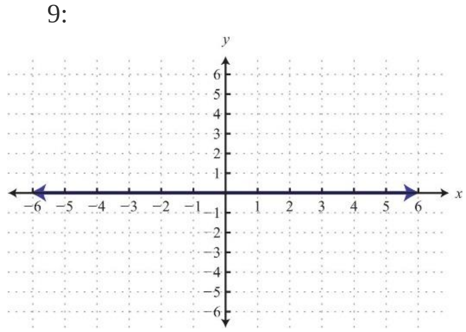

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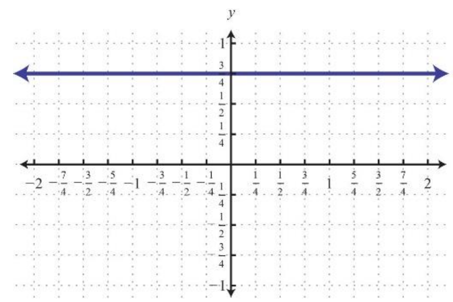

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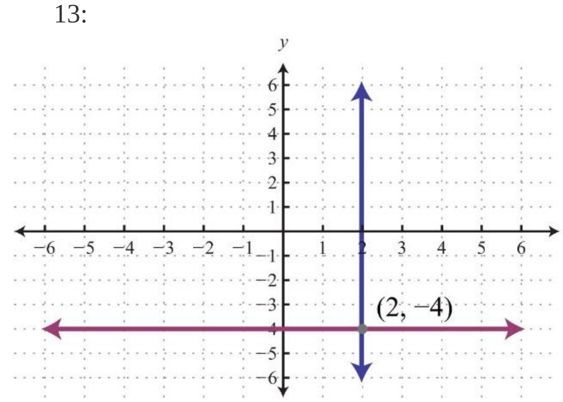


2.1 (Part 3) Answers – Graph Using Intercepts

1: y-intercept: $(0, -3)$; x-intercept: $(4, 0)$

3: y-intercept: $(0, -3)$; x-intercept: none

5: y-intercept: $(0, 0)$; x-intercept: $(0, 0)$

7: x-intercept: $(4, 0)$; y-intercept: $(0, -5)$

9: x-intercept: $(3, 0)$; y-intercept: $(0, -3)$

11: x-intercept: $(1/3, 0)$; y-intercept: $(0, -1/4)$

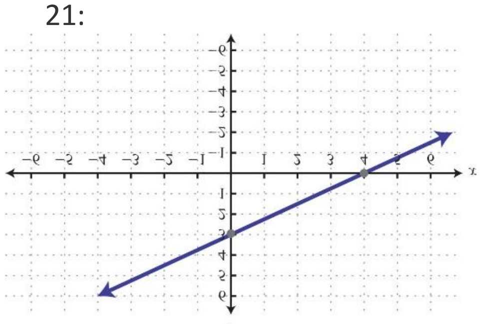
13: x-intercept: $(4, 0)$; y-intercept: $(0, -3)$

15: x-intercept: none; y-intercept: $(0, 6)$

17: x-intercept: $(2, 0)$; y-intercept: none

19: x-intercept: $(-b/m, 0)$; y-intercept: $(0, b)$

21:



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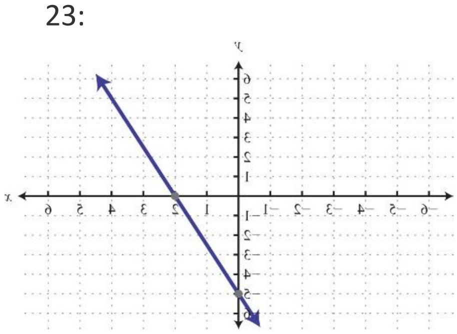

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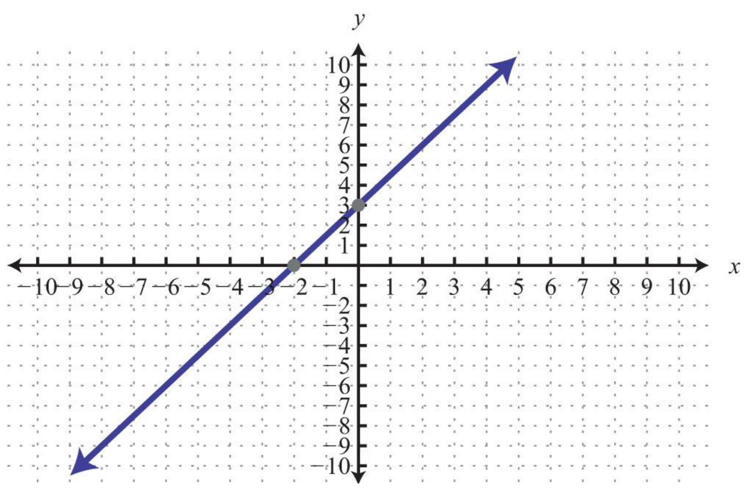
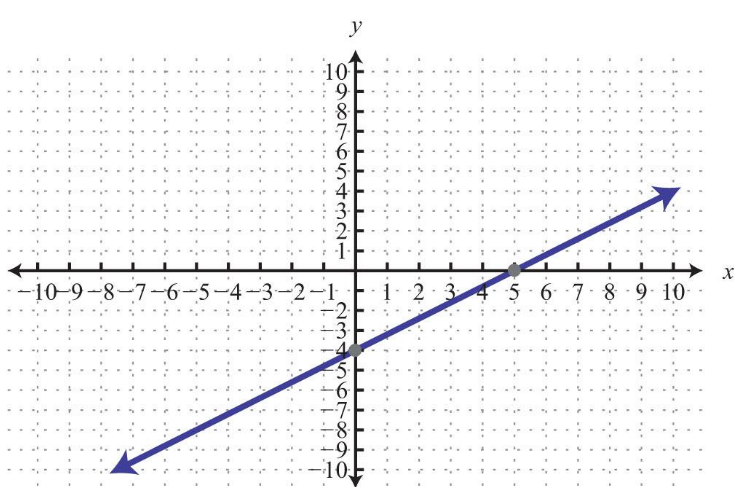
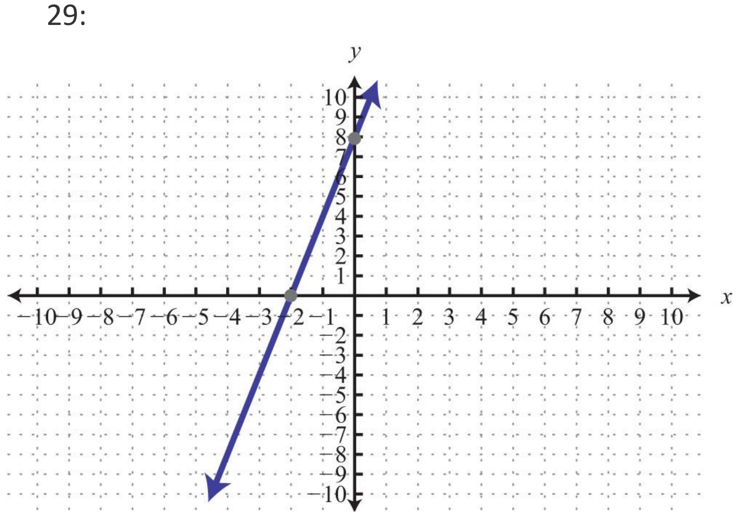

y
x
10
9
8
7
6
5
4
3
2
1
−10 −9 −8 −7 −6 −5 −4 −3 −2 −1
1 2 3 4 5 6 7 8 9 10
−2
−3
−4
−5
−6
−7
−8
−9
−10

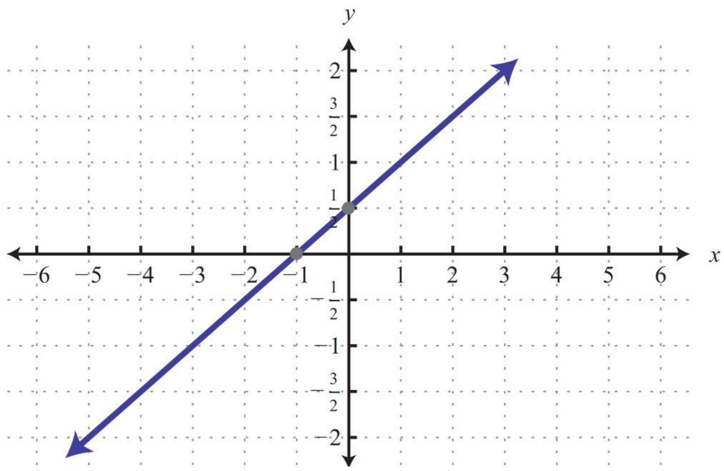
27:


y
x
10
9
8
7
6
5
4
3
2
1
−10 −9 −8 −7 −6 −5 −4 −3 −2 −1
1 2 3 4 5 6 7 8 9 10
−2
−3
−4
−5
−6
−7
−8
−9
−10

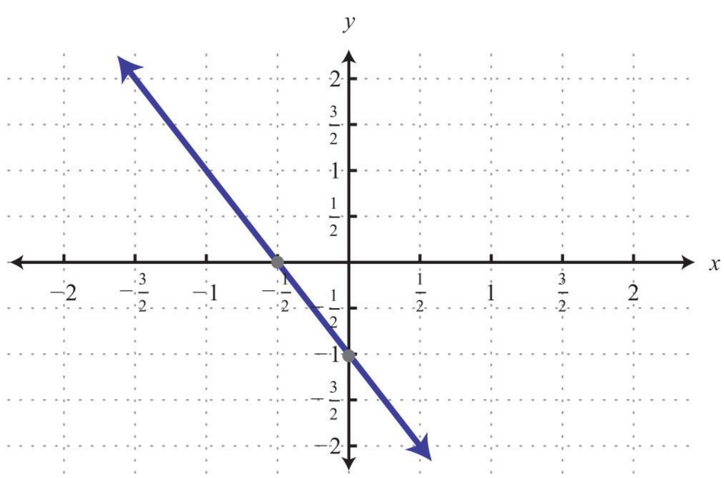
29:


y
x
10
9
8
7
6
5
4
3
2
1
−10 −9 −8 −7 −6 −5 −4 −3 −2 −1
1 2 3 4 5 6 7 8 9 10
−2
−3
−4
−5
−6
−7
−8
−9
−10

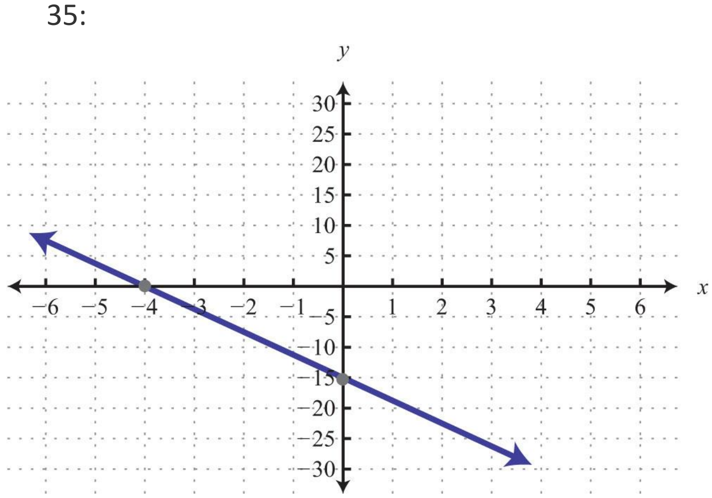
31:


y
x
2
3/2
1
1/2
−1/2
−1
−3/2
−2
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

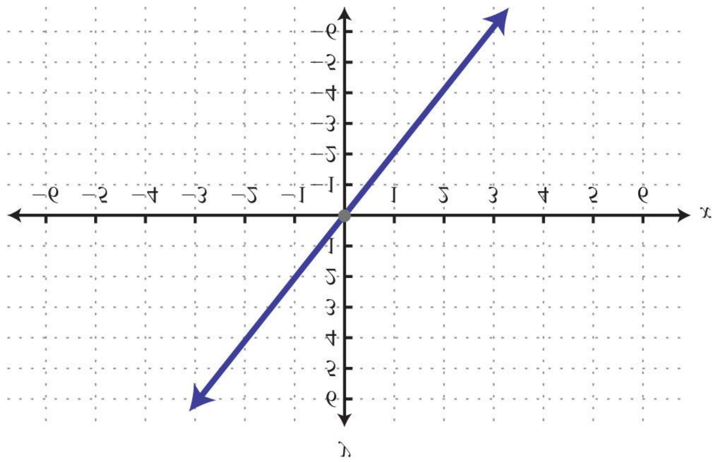
33:


y
x
2
3/2
1
1/2
−1/2
−1
−3/2
−2
−2 −3/2 −1 −1/2 1/2 1 3/2 2

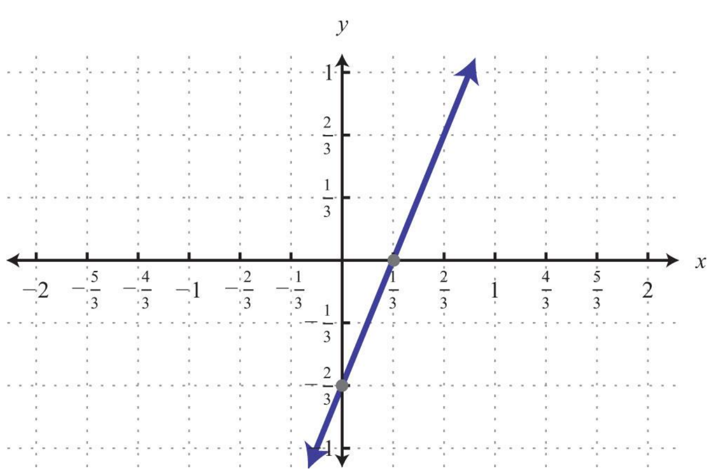
35:


y
x
30
25
20
15
10
5
−5
−10
−15
−20
−25
−30
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

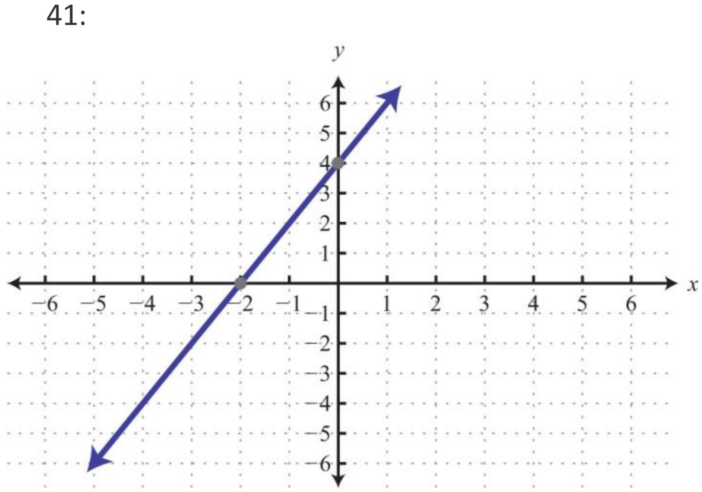
37:



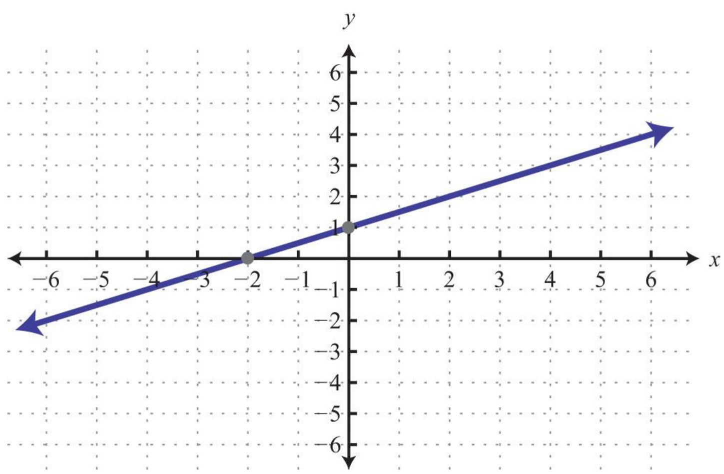
39:


y
x
1
$\frac{2}{3}$
$\frac{1}{3}$
$-\frac{1}{3}$
$-\frac{2}{3}$
-2
$-\frac{5}{3}$
$-\frac{4}{3}$
-1
$-\frac{2}{3}$
$-\frac{1}{3}$
$\frac{1}{3}$
$\frac{2}{3}$
1
$\frac{4}{3}$
$\frac{5}{3}$
2

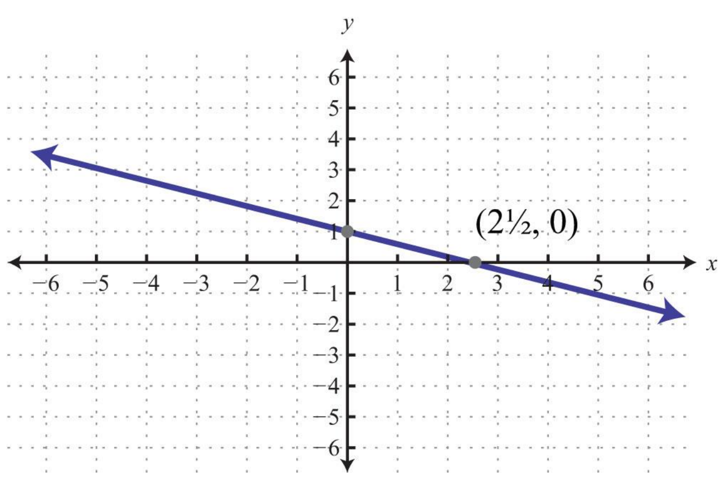
41:


y
x
6
5
4
3
2
1
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6
−1
−2
−3
−4
−5
−6

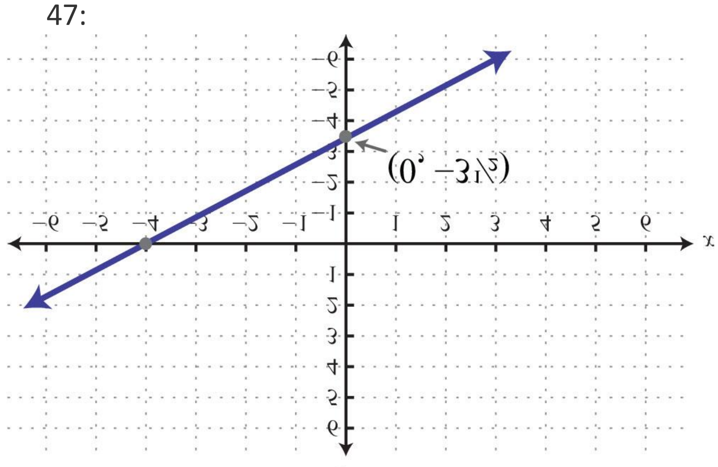
43:



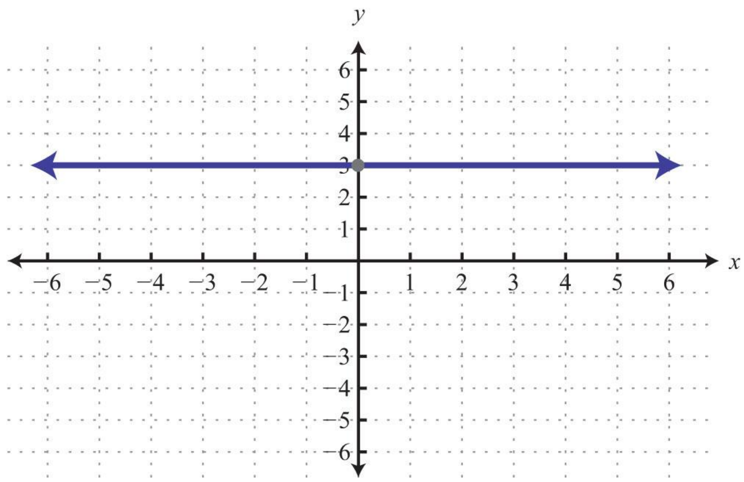
45:



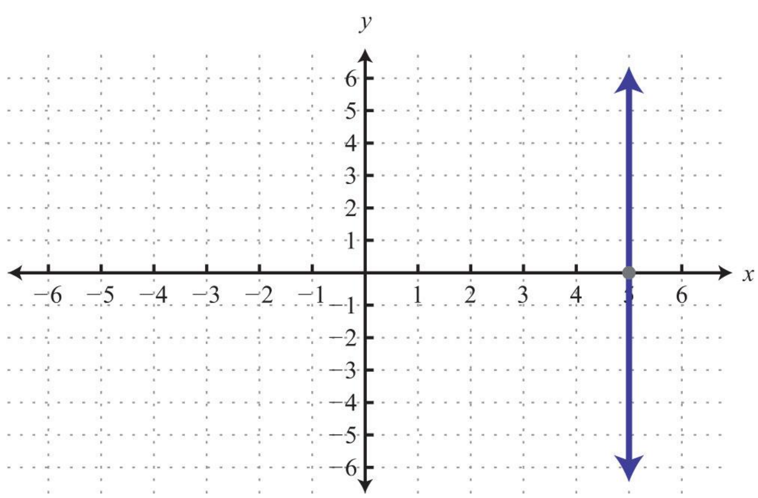
47:



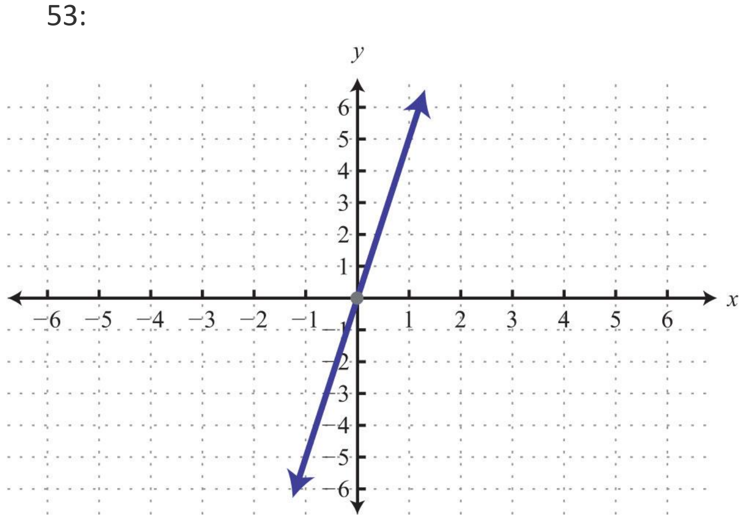
49:


y
x
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6

51:


y
x
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−6
−5
−4
−3
−2
−1
1
2
3
4
6

53:


y
x
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6

55: x-intercepts: $(-3, 0)$, $(3, 0)$; y-intercept: $(0, -3)$

57: x-intercepts: $(-4, 0)$, $(0, 0)$; y-intercept: $(0, 0)$

59: x-intercepts: $(-2, 0)$, $(2, 0)$; y-intercept: $(0, -1)$

61: x-intercepts: $(-3, 0)$, $(0, 0)$, $(2, 0)$; y-intercept: $(0, 0)$

63: x-intercepts: $(-4, 0)$, $(4, 0)$; y-intercepts: $(0, -4)$, $(0, 4)$

2.2

2.2 Answers - Slope

1) $\frac{3}{2}$

2) 5

3) Undefined

4) $-\frac{1}{2}$

5) $\frac{5}{6}$

6) $-\frac{2}{3}$

7) -1

8) $\frac{5}{4}$

9) -1

10) 0

11) Undefined

12) $\frac{16}{7}$

13) $-\frac{17}{31}$

14) $-\frac{3}{2}$

15) $\frac{4}{3}$

16) $-\frac{7}{17}$

17) 0

18) $\frac{5}{11}$

19) $\frac{1}{2}$

20) $\frac{1}{16}$

21) $-\frac{11}{2}$

22) $-\frac{12}{31}$

23) Undefined

24) $\frac{24}{11}$

25) $-\frac{26}{27}$

26) $-\frac{19}{10}$

27) $-\frac{1}{3}$

28) $\frac{1}{16}$

29) $-\frac{7}{13}$

30) $\frac{2}{7}$

31) -5

32) 2

33) -8

34) 3

35) -5

36) 6

37) -4

38) 1

39) 2

40) 1

2.3

2.3 Answers - Slope-Intercept

1) $y=2x+5$

2) $y=-6x+4$

3) $y=x-4$

4) $y=-x-2$

5) $y=-\frac{3}{4}x-1$

6) $y=-\frac{1}{4}x+3$

7) $y=\frac{1}{3}x+1$

8) $y=\frac{2}{5}x+5$

9) $y=-x+5$

10) $y=-\frac{7}{2}x-5$

11) $y=x-1$

12) $y=-\frac{5}{3}x-3$

13) $y=-4x$

14) $y=-\frac{3}{4}x+2$

15) $y=-\frac{1}{10}x-\frac{37}{10}$

16) $y=\frac{1}{10}x-\frac{3}{10}$

17) $y=-2x-1$

18) $y=\frac{6}{11}x+\frac{70}{11}$

19) $y=\frac{7}{3}x-8$

20) $y=-\frac{4}{7}x+4$

21) $x=-8$

22) $y=\frac{1}{7}x+6$

23) $y=-x-1$

24) $y=\frac{5}{2}x$

25) $y=4x$

26) $y=-\frac{2}{3}x+1$

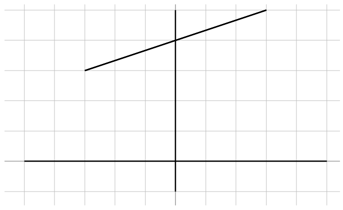
27) $y=-4x+3$

28) $x=4$

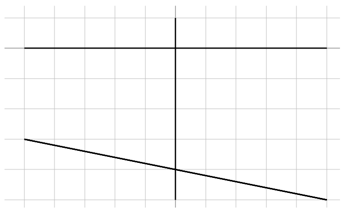
29) $y=-\frac{1}{2}x+1$

30) $y=\frac{6}{5}x+4$

31)



32)



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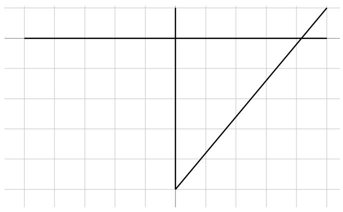

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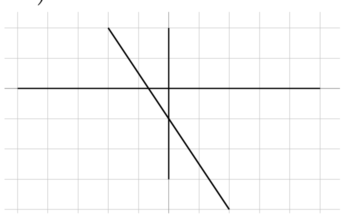

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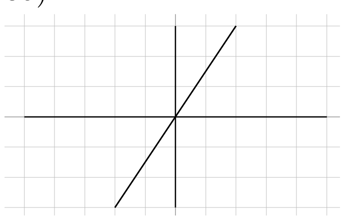

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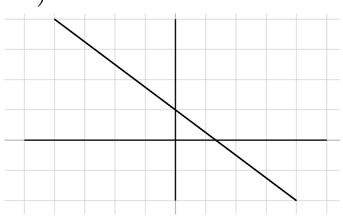
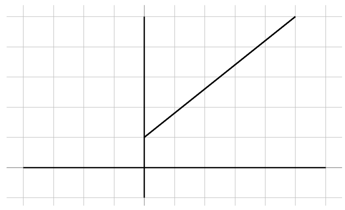
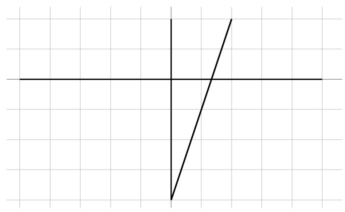


37)

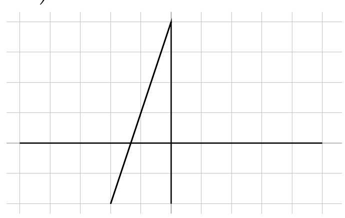
38)



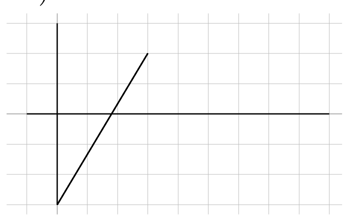
39)



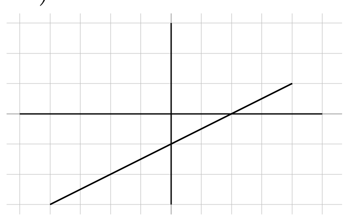
40)



41)



42)



2.4

2.4 Answers - Point-Slope Form

1) $x=2$
2) $x=1$
3) $y-2=\frac{1}{2}(x-2)$
4) $y-1=-\frac{1}{2}(x-2)$
5) $y+5=9(x+1)$
6) $y+2=-2(x-2)$
7) $y-1=\frac{3}{4}(x+4)$
8) $y+3=-2(x-4)$
9) $y+2=-3x$
10) $y-1=4(x+1)$
11) $y+5=-\frac{1}{4}x$
12) $y-2=-\frac{5}{4}x$
13) $y+3=\frac{1}{5}(x+5)$
14) $y+4=-\frac{2}{3}(x+1)$
15) $y-4=-\frac{5}{4}(x+1)$
16) $y+4=-\frac{3}{2}x(x-1)$
17) $y=2x-3$
18) $y=-2x+2$
19) $y=-\frac{3}{5}x+2$
20) $y=-\frac{2}{3}x-\frac{10}{3}$
21) $y=\frac{1}{2}x+3$
22) $y=-\frac{7}{4}x+4$
23) $y=-\frac{3}{2}x+4$
24) $y=-\frac{5}{2}x-5$
25) $y=-\frac{2}{5}x-5$
26) $y=\frac{7}{3}x-4$
27) $y=x-4$
28) $y=-3$
29) $x=-3$
30) $y=2x-1$
31) $y=-\frac{1}{2}x$
32) $y=\frac{6}{5}x-3$
33) $y-3=-2(x+4)$
34) $y=3$
35) $y-1=\frac{1}{8}(x-5)$
36) $y-5=-\frac{1}{8}(x+4)$
37) $y+2=\frac{3}{2}(x+4)$
38) $y-1=\frac{3}{8}(x+4)$
39) $y-5=\frac{1}{4}(x-3)$
40) $y+4=-(x+1)$
41) $y+3=-\frac{8}{7}(x-3)$
42) $y+5=-\frac{1}{4}(x+1)$
43) $y=-\frac{3}{4}x-\frac{11}{4}$
44) $y=-\frac{1}{10}x-\frac{3}{2}$
45) $y=-\frac{8}{7}x-\frac{5}{7}$
46) $y=\frac{1}{2}x-\frac{3}{2}$
47) $y=-x+5$
48) $y=\frac{1}{3}x+1$
49) $y=-x+2$
50) $y=x+2$
51) $y=4x+3$
52) $y=\frac{3}{7}x+\frac{6}{7}$

2.5

2.5 Answers - Parallel and Perpendicular Lines

1) 2
2) $-\frac{2}{3}$
3) 4
4) $-\frac{10}{3}$
5) 1
6) $\frac{6}{5}$
7) -7
8) $-\frac{3}{4}$
9) 0
10) 2
11) 3
12) $-\frac{5}{4}$
13) -3
14) $-\frac{1}{3}$
15) 2
16) $-\frac{3}{8}$
17) $x=2$
18) $y-2=\frac{7}{5}(x-5)$
19) $y-4=\frac{9}{2}(x-3)$
20) $y+1=-\frac{3}{4}(x-1)$
21) $y-3=\frac{7}{5}(x-2)$
22) $y-3=-3(x+1)$

23) $x=4$

24) $y-4=\frac{7}{5}(x-1)$

25) $y+5=-(x-1)$

26) $y+2=-2(x-1)$

27) $y-2=\frac{1}{5}(x-5)$

28) $y-3=-(x-1)$

29) $y-2=-\frac{1}{4}(x-4)$

30) $y+5=\frac{7}{3}(x+3)$

31) $y+2=-3(x-2)$

32) $y-5=-\frac{1}{2}(x+2)$

33) $y=-2x+5$

34) $y=\frac{3}{5}x+5$

35) $y=-\frac{4}{3}x-3$

36) $y=-\frac{5}{4}x-5$

37) $y=-\frac{1}{2}x-3$

38) $y=\frac{5}{2}x-2$

39) $y=-\frac{1}{2}x-2$

40) $y=\frac{3}{5}x-1$

41) $y=x-1$

42) $y=2x+1$

43) $y=2$

44) $y=-\frac{2}{5}x+1$

45) $y=-x+3$

46) $y=-\frac{5}{2}x+2$

47) $y=-2x+5$

48) $y=\frac{3}{4}x+4$

Answers - Chapter 3

3.1

Answers - Solve and Graph Inequalities

1) $(-5,\infty)$

2) $(-4,\infty)$

3) $(-\infty,-2]$

4) $(-\infty,1]$

5) $(-\infty,5]$

6) $(-5,\infty)$

7) $m<-2$

8) $m\leqslant 1$

9) $x\geqslant 5$

10) $a\leqslant -5$

11) $b>-2$

12) $x>1$

13) $x\geqslant 110\colon [110,\infty)$

14) $n\geqslant -26\colon [-26,\infty)$

15) $r<1\colon (-\infty,1)$

16) $m\leqslant -6\colon (-\infty,-6]$

17) $n\geqslant -6\colon [-6,\infty)$

18) $x<6\colon (-\infty,6)$

19) $a<12\colon (-\infty,12)$

20) $v\geqslant 1\colon [1,\infty)$

21) $x\geqslant 11\colon [11,\infty)$

22) $x\leqslant -18\colon (-\infty,-18]$

23) $k>19\colon (19,\infty)$

24) $n\leqslant -10\colon (-\infty,-10]$

25) $p<-1\colon (-\infty,-1)$

26) $x\leqslant 20\colon (-\infty,20]$

27) $m\geqslant 2\colon [2,\infty)$

28) $n\leqslant 5\colon (-\infty,5]$

29) $r>8\colon (8,\infty)$

30) $x\leqslant -3\colon (-\infty,-3]$

31) $b>1\colon (1,\infty)$

32) $n\geqslant 0\colon [0,\infty)$

33) $v<0\colon (-\infty,0)$

34) $x>2\colon (2,\infty)$

35) No solution: $\oslash$

36) $n > 1 : (1, \infty)$

37) {All real numbers.} : $\mathbb{R}$

38) $p \leqslant 3 : (-\infty, 3]$

3.2

Answers - Compound Inequalities

1) $n \leqslant -9$ or $n \geqslant 2 : (-\infty, -9] \bigcup [2, \infty)$

2) $m \geqslant -4$ or $m < -5$: $(-\infty, -5) \bigcup [-4, \infty)$

3) $x \geqslant 5$ or $x < -5$: $(-\infty, -5) \bigcup [5, \infty)$

4) $r > 0$ or $r < -7$: $(-\infty - 7), \bigcup (0, \infty)$

5) $x < -7 : (-\infty, -7)$

6) $n < -7$ or $n > 8$: $(-\infty - 7), \bigcup (8, \infty)$

7) $-8 < v < 3 : (-8, 3)$

8) $-7 < x < 4 : (-7, 4)$

9) $b < 5 : (-\infty, 5)$

10) $-2 \leqslant n \leqslant 6 : [-2, 6]$

11) $-7 \leqslant a \leqslant 6 : [-7, 6]$

12) $v \geqslant 6 : [6, \infty)$

13) $-6 \leqslant x \leqslant -2 : [-6, -2]$

14) $-9 \leqslant x \leqslant 0 : [-9, 0]$

15) $3 < k \leqslant 4 : (3, 4]$

16) $-2 \leqslant n \leqslant 4 : [-2, 4]$

17) $-2 < x < 2 : (-2, 2)$

18) No solution : $\oslash$

19) $-1 \leqslant m < 4 : [-1, 4)$

20) $r > 8$ or $r < -6$: : $(-\infty, -6) \bigcup (8, \infty)$

21) No solution : $\oslash$

22) $x \leqslant 0$ or $x > 8$: $(-\infty, 0] \bigcup (8, \infty)$

23) No solution : $\oslash$

24) $n \geqslant 5$ or $n < 1$: $(-\infty, 1) \bigcup [5, \infty)$

25) $5 \leqslant x < 19$: $[5, 19)$

26) $n < -14$ or $n \geqslant 17$: $(-\infty, -14) \bigcup [17, \infty)$

27) $1 \leqslant v \leqslant 8$: $[1, 8]$

28) $a \leqslant 1$ or $a \geqslant 19$: $(-\infty, 1] \bigcup [19, \infty)$

29) $k \geqslant 2$ or $k < -20$: $(-\infty, -20) \bigcup [2, \infty)$

30) $\{$All real numbers.$\}$: $\mathbb{R}$

31) $-1 < x \leqslant 1$: $(-1, 1]$

32) $m > 4$ or $m \leqslant -1$: $(-\infty, -1] \bigcup (4, \infty)$

3.3

Answers - Absolute Value Inequalities

1) $-3, 3$

2) $-8, 8$

3) $-3, 3$

4) $-7, 1$

5) $-4, 8$

6) $-4, 20$

7) $-2, 4$

8) $-7, 1$

9) $-\frac{7}{3}, \frac{11}{3}$

10) $-7, 2$

11) $-3, 5$

12) $0, 4$

13) $1, 4$

14) $(-\infty, 5) \bigcup (5, \infty)$

15) $(-\infty, -\frac{5}{3}] \bigcup \left[\frac{5}{3}, \infty\right)$

16) $(-\infty, -1] \bigcup [9, \infty)$

17) $(-\infty, -6) \bigcup (0, \infty)$

18) $(-\infty, -1) \bigcup (5, \infty)$

19) $(-\infty, \frac{2}{3}) \bigcup (\frac{8}{3}, \infty)$

20) $(-\infty, 0) \bigcup (4, \infty)$

21) $(-\infty, -1] \bigcup [3, \infty)$

22) $[-\frac{4}{3}, 2]$

23) $(-\infty, -4] \bigcup [14, \infty)$

24) $(-\infty, -\frac{5}{2}] \bigcup [-\frac{3}{2}, \infty)$

25) $[1, 3]$

26) $[\frac{1}{2}, 1]$

27) $(-\infty, -4) \bigcup (-3, \infty)$

28) $[3, 7]$

29) $[1, \frac{3}{2}]$

30) $[-2, -\frac{4}{3}]$

31) $(-\infty, \frac{3}{2}) \bigcup (\frac{5}{2}, \infty)$

32) $(-\infty, -\frac{1}{2}) \bigcup (1, \infty)$

33) $[2, 4]$

34) $[-3, -2]$

Answers - Chapter 4

4.1

Answers - Graphing

1) $(-1,2)$
2) $(-4,3)$
3) $(-1,-3)$
4) $(-3,1)$
5) No Solution
6) $(-2,-2)$
7) $(-3,1)$
8) $(4,4)$
9) $(-3,-1)$
10) No Solution
11) $(3,-4)$
12) $(4,-4)$
13) $(1,-3)$
14) $(-1,3)$
15) $(3,-4)$
16) No Solution
17) $(2,-2)$
18) $(4,1)$
19) $(-3,4)$
20) $(2,-1)$
21) $(3,2)$
22) $(-4,-4)$
23) $(-1,-1)$
24) $(2,3)$
25) $(-1,-2)$
26) $(-4,-3)$
27) No Solution
28) $(-3,1)$
29) $(4,-2)$
30) $(1,4)$

4.2

Answers - Substitution

1) $(1,-3)$
2) $(-3,2)$
3) $(-2,-5)$
4) $(0,3)$
5) $(-1,-2)$
6) $(-7,-8)$
7) $(1,5)$
8) $(-4,-1)$
9) $(3,3)$
10) $(4,4)$
11) $(2,6)$
12) $(-3,3)$
13) $(-2,-6)$
14) $(0,2)$
15) $(1,-5)$
16) $(-1,0)$
17) $(-1,8)$
18) $(3,7)$
19) $(2,3)$
20) $(8,-8)$
21) $(1,7)$
22) $(1,7)$
23) $(-3,-2)$
24) $(1,-3)$
25) $(1,3)$
26) $(2,1)$
27) $(-2,8)$
28) $(-4,3)$
29) $(4,-3)$
30) $(-1,5)$
31) $(0,2)$
32) $(0,-7)$
33) $(0,3)$
34) $(1,-4)$
35) $(4,-2)$
36) $(8,-3)$
37) $(2,0)$
38) $(2,5)$
39) $(-4,8)$
40) $(2,3)$

4.3

Answers - Addition/Elimination

1) $(-2,4)$
2) $(2,4)$
3) No solution
4) Infinite number of solutions
5) No solution
6) Infinite number of solutions
7) No solution
8) $(2,-2)$
9) $(-3,-5)$
10) $(-3,6)$
11) $(-2,-9)$
12) $(1,-2)$
13) $(0,4)$
14) $(-1,0)$
15) $(8,2)$
16) $(0,3)$
17) $(4,6)$
18) $(-6,-8)$
19) $(-2,3)$
20) $(1,2)$
21) $(0,-4)$
22) $(0,1)$
23) $(-2,0)$
24) $(2,-2)$
25) $(-1,-2)$
26) $(-3,0)$
27) $(-1,-3)$
28) $(-3,0)$
29) $(-8,9)$
30) $(1,2)$
31) $(-2,1)$
32) $(-1,1)$
33) $(0,0)$
34) Infinite number of solutions

4.4

Answers - Three Variables

1) $(1,-1,2)$
2) $(5,-3,2)$
3) $(2,3,-2)$
4) $(3,-2,1)$
5) $(-2,-1,4)$
6) $(-3,2,1)$
7) $(1,2,3)$
8) $\propto$ solutions
9) $(0,0,0)$
10) $\propto$ solutions
11) $(19,0,-13)$
12) $\propto$ solutions
13) $(0,0,0)$
14) $\propto$ solutions
15) $(2,\frac{1}{2},-2)$
16) $\propto$ solutions
17) $(-1,2,3)$
18) $(-1,2,-2)$
19) $(0,2,1)$
20) no solution
21) $(10,2,3)$
22) no solution
23) $(2,3,1)$
24) $\propto$ solutions
25) no solutions
26) $(1,2,4)$
27) $(-25,18,-25)$
28) $(\frac{2}{7},\frac{3}{7},-\frac{2}{7})$
29) $(1,-3,-2,-1)$
30) $(7,4,5,6)$
31) $(1,-2,4,-1)$
32) $(-3,-1,0,4)$

4.5 Answers – Solving Systems with Matrices

7. $\left[\begin{array}{cc|c} 0 & 16 & 4 \\ 9 & -1 & 2 \end{array}\right]$ **9.** $\left[\begin{array}{ccc|c} 1 & 5 & 8 & 19 \\ 12 & 3 & 0 & 4 \\ 3 & 4 & 9 & -7 \end{array}\right]$

11. $\begin{aligned} -2x + 5y &= 5 \\ 6x - 18y &= 26 \end{aligned}$

13. $\begin{aligned} 3x + 2y &= 13 \\ -x - 9y + 4z &= 53 \\ 8x + 5y + 7z &= 80 \end{aligned}$

15. $\begin{aligned} 4x + 5y - 2z &= 12 \\ y + 58z &= 2 \\ 8x + 7y - 3z &= -5 \end{aligned}$

17. No solutions

19. $(-1, -2)$

21. $(6, 7)$

23. $(3, 2)$ **25.** $\left(\frac{1}{5}, \frac{1}{2}\right)$ **27.** $\left(x, \frac{4}{15}(5x + 1)\right)$ **29.** $(3, 4)$

31. $\left(\frac{196}{39}, -\frac{5}{13}\right)$ **33.** $(31, -42, 87)$ **35.** $\left(\frac{21}{40}, \frac{1}{20}, \frac{9}{8}\right)$

37. $\left(\frac{18}{13}, \frac{15}{13}, -\frac{15}{13}\right)$ **39.** $\left(x, y, \frac{1}{2} - x - \frac{3}{2}y\right)$

41. $\left(x, -\frac{x}{2}, -1\right)$ **43.** $(125, -25, 0)$ **45.** $(8, 1, -2)$

47. $(1, 2, 3)$ **49.** $\left(-4z + \frac{17}{7}, 3z - \frac{10}{7}, z\right)$

51. No solutions exist.

4.6 Answers – Applications of Systems

1: The integers are 22 and 32.

3: The integers are 8 and 24.

5: The integers are 16 and 58.

7: The two numbers are −1/4 and 1/4.

9: The smaller number is −4 and the larger is 2/3.

11: $5,700 at 3% and $1,300 at 7%

13: $1,300 at 6% and $500 at 3%

15: $8,500 at 7% and $1,500 at 4%

17: $1,400 at 4% and $3,800 at 6%

29: 1.5 pounds of the 10% cashew mix and 0.5 pounds of the 30% cashew mix

31: 12 ounces of cleaning fluid

33: 4.4 ounces of the 50% ammonia solution and 3.6 ounces of the 10% ammonia solution

35: 4 gallons

37: The airplane flew 3 hours with the wind and 4 hours against the wind.

39: 3.5 hours

19: 12 tens and 11 twenties

21: 52 dimes and 28 nickels

23: Shirts: \$4.50; shorts: \$9.50

25: 6 gallons of each

27: 32 ounces of the 1% saline solution and 8 ounces of the 2% saline solution

41: 4.5 hours

43: Boat: 9 miles per hour; current: 3 miles per hour

45: 1.5 miles per hour

Answers - Chapter 5

5.1

Answers to Exponent Properties

1) 4^9

2) 4^7

3) 2^4

4) 3^6

5) $12m^2n$

6) $12x^3$

7) $8m^6n^3$

8) x^3y^6

9) 3^{12}

10) 4^{12}

11) 4^8

12) 3^6

13) $4u^6v^4$

14) x^3y^3

15) $16a^{16}$

16) $16x^4y^4$

17) 4^2

18) 3^4

19) 3

20) 3^3

21) m^2

22) $\frac{xy^3}{4}$

23) $\frac{4x^2y}{3}$

24) $\frac{y^2}{4}$

25) $4x^{10}y^{14}$

26) $8u^{18}v^6$

27) $2x^{17}y^{16}$

28) $3uv$

29) $\frac{x^2y}{6}$

30) $\frac{4a^2}{3}$

31) 64

32) $2a$

33) $\frac{y^3}{512x^{24}}$

34) $\frac{y^5x^2}{2}$

35) $64m^{12}n^{12}$

36) $\frac{n^{10}}{2m}$

37) $2x^2y$

38) $2y^2$

39) $2q^7r^8p$

40) $4x^2y^4z^2$

41) $x^4y^{16}z^4$

42) $256q^4r^8$

43) $4y^4z$

5.2

Answers to Negative Exponents

1) $32x^8y^{10}$

2) $\frac{32b^{13}}{a^2}$

3) $\frac{2a^{15}}{b^{11}}$

4) $2x^3y^2$

5) $16x^4y^8$

6) 1

7) $y^{16}x^5$

8) $\frac{32}{m^5n^{15}}$

9) $\frac{2}{9y}$

10) $\frac{y^5}{2x^7}$

11) $\frac{1}{y^2x^3}$

12) $\frac{y^8x^5}{4}$

13) $\frac{u}{4v^6}$

14) $\frac{x^7y^2}{2}$

15) $\frac{u^2}{12v^5}$

16) $\frac{y}{2x^4}$

17) $\frac{2}{y^7}$

18) $\frac{a^{16}}{2b}$

19) $16a^{12}b^{12}$

20) $\frac{y^8x^4}{4}$

21) $\frac{1}{8m^4n^7}$

22) $2x^{16}y^2$

23) $16n^6m^4$

24) $\frac{2x}{y^3}$

25) $\frac{1}{x^{15}y}$

26) $4y^4$

27) $\frac{u}{2v}$

28) $4y^5$

29) 8

30) $\frac{1}{2u^3v^5}$

31) $2y^5x^4$

32) $\frac{a^3}{2b^3}$

33) $\frac{1}{x^2y^{11}z}$

34) $\frac{a^2}{8c^{10}b^{12}}$

35) $\frac{1}{h^3kj^6}$

36) $\frac{x^{30}z^6}{16y^4}$

37) $\frac{2b^{14}}{a^{12}c^7}$

38) $\frac{m^{14}q^8}{4p^4}$

39) $\frac{x^2}{y^4z^4}$

40) $\frac{mn^7}{p^5}$

5.3

Answers to Scientific Notation

1) 8.85×10^2

2) 7.44×10^{-4}

3) 8.1×10^{-2}

4) 1.09×10^0

5) 3.9×10^{-2}

6) 1.5×10^4

7) 870000

8) 256

9) 0.0009

10) 50000

11) 2

12) 0.00006

13) 1.4×10^{-3}

14) 1.76×10^{-10}

15) 1.662×10^{-6}

16) 5.018×10^6

17) 1.56×10^{-3}

18) 4.353×10^8

19) 1.815×10^4

20) 9.836×10^{-1}

21) 5.541×10^{-5}

22) 6.375×10^{-4}

23) 3.025×10^{-9}

24) 1.177×10^{-16}

25) 2.887×10^{-6}

26) 6.351×10^{-21}

27) 2.405×10^{-20}

28) 2.91×10^{-2}

29) 1.196×10^{-2}

30) 1.2×10^7

31) 2.196×10^{-2}

32) 2.52×10^3

33) 1.715×10^{14}

34) 8.404×10^1

35) 1.149×10^6

36) 3.939×10^9

37) 4.6×10^2

38) 7.474×10^3

39) 3.692×10^{-7}

40) 1.372×10^3

41) 1.034×10^6

42) 1.2×10^6

5.4

Answers to Introduction to Polynomials

1) 3
2) 7
3) -10
4) -6
5) -7
6) 8
7) 5
8) -1
9) 12
10) -1
11) $3p^4 - 3p$
12) $-m^3 + 12m^2$
13) $-n^3 + 10n^2$
14) $8x^3 + 8x^2$
15) $5n^4 + 5n$
16) $2v^4 + 6$
17) $13p^3$
18) $-3x$
19) $3n^3 + 8$
20) $x^4 + 9x^2 - 5$
21) $2b^4 + 2b + 10$
22) $-3r^4 + 12r^2 - 1$
23) $-5x^4 + 14x^3 - 1$
24) $5n^4 - 4n + 7$
25) $7a^4 - 3a^2 - 2a$
26) $12v^3 + 3v + 3$
27) $p^2 + 4p - 6$
28) $3m^4 - 2m + 6$
29) $5b^3 + 12b^2 + 5$
30) $-15n^4 + 4n - 6$
31) $n^3 - 5n^2 + 3$
32) $-6x^4 + 13x^3$
33) $-12n^4 + n^2 + 7$
34) $9x^2 + 10x^2$
35) $r^4 - 3r^3 + 7r^2 + 1$
36) $10x^3 - 6x^2 + 3x - 8$
37) $9n^4 + 2n^3 + 6n^2$
38) $2b^4 - b^3 + 4b^2 + 4b$
39) $-3b^4 + 13b^3 - 7b^2 - 11b + 19$
40) $12n^4 - n^3 - 6n^2 + 10$
41) $2x^4 - x^3 - 4x + 2$
42) $3x^4 + 9x^2 + 4x$

5.5

Answers to Multiply Polynomials

1) $6p - 42$
2) $32k^2 + 16k$
3) $12x + 6$
4) $18n^3 + 21n^2$
5) $20m^5 + 20m^4$
6) $12r - 21$
7) $32n^2 + 80n + 48$
8) $2x^2 - 7x - 4$
9) $56b^2 - 19b - 15$
10) $4r^2 + 40r + 64$
11) $8x^2 + 22x + 15$
12) $7n^2 + 43n - 42$
13) $15v^2 - 26v + 8$
14) $6a^2 - 44a - 32$
15) $24x^2 - 22x - 7$
16) $20x^2 - 29x + 6$
17) $30x^2 - 14xy - 4y^2$
18) $16u^2 + 10uv - 21v^2$
19) $3x^2 + 13xy + 12y^2$
20) $40u^2 - 34uv - 48v^2$
21) $56x^2 + 61xy + 15y^2$
22) $5a^2 - 7ab - 24b^2$
23) $6r^3 - 43r^2 + 12r - 35$

24) $16x^3+44x^2+44x+40$

25) $12n^3-20n^2+38n-20$

26) $8b^3-4b^2-4b-12$

27) $36x^3-24x^2y+3xy^2+12y^3$

28) $21m^3+4m^2n-8n^3$

29) $48n^4-16n^3+64n^2-6n+36$

30) $14a^4+30a^3-13a^2-12a+3$

31) $15k^4+24k^3+48k^2+27k+18$

32) $42u^4+76u^3v+17u^2v^2-18v^4$

33) $18x^2-15x-12$

34) $10x^2-55x+60$

35) $24x^2-18x-15$

36) $16x^2-44x-12$

37) $7x^2-49x+70$

38) $40x^2-10x-5$

39) $96x^2-6$

40) $36x^2+108x+81$

5.6

Answers to Multiply Special Products

1) x^2-64

2) a^2-16

3) $1-9p^2$

4) x^2-9

5) $1-49n^2$

6) $64m^2-25$

7) $25n^2-64$

8) $4r^2-9$

9) $16x^2-64$

10) b^2-49

11) $16y^2-x^2$

12) $49a^2-49b^2$

13) $16m^2-64n^2$

14) $9y^2-9x^2$

15) $36x^2-4y^2$

16) $1+10n+25n^2$

17) $a^2+10a+25$

18) $v^2+8v+16$

19) $x^2-16x+64$

20) $1-12n+36n^2$

21) $p^2+14p+49$

22) $49k^2-98k+49$

23) $49-70n+25n^2$

24) $16x^2-40x+25$

25) $25m^2-80m+64$

26) $9a^2+18ab+9b^2$

27) $25x^2+70xy+49y^2$

28) $16m^2-8mn+n^2$

29) $4x^2+8xy+4y^2$

30) $64x^2+80xy+25y^2$

31) $25+20r+4r^2$

32) $m^2-14m+49$

33) $4+20x+25x^2$

34) $64n^2-49$

35) $16v^2-49$

36) b^2-16

37) n^2-25

38) $49x^2+98x+49$

39) $16k^2+16k+4$

40) $9a^2-64$

5.7

Answers to Divide Polynomials

1) $5x+\frac{1}{4}+\frac{1}{2x}$

2) $\frac{5x^3}{9}+5x^2+\frac{4x}{9}$

3) $2n^3+\frac{n^2}{10}+4n$

4) $\frac{3k^2}{8}+\frac{k}{2}+\frac{1}{4}$

5) $2x^3+4x^2+\frac{x}{2}$

6) $\frac{5p^3}{4}+4p^2+4p$

7) $n^2+5n+\frac{1}{5}$

8) $\frac{m^2}{3}+2m+3$

9) $x-10+\frac{9}{x+8}$

10) $r+6+\frac{1}{r-9}$

11) $n+8-\frac{8}{n+5}$

12) $b-3-\frac{5}{b-7}$

13) $v+8-\frac{9}{v-10}$

14) $x-3-\frac{5}{x+7}$

15) $a+4-\frac{6}{a-8}$

16) $x-6-\frac{2}{x-4}$

17) $5p+4+\frac{3}{9p+4}$

18) $8k-9-\frac{1}{3k-1}$

19) $x-3+\frac{3}{10x-2}$

20) $n+3+\frac{3}{n+4}$

21) $r-1+\frac{2}{4x+3}$

22) $m+4+\frac{1}{m-1}$

23) $n+2$

24) $x-4+\frac{4}{2x+3}$

25) $9b+5-\frac{5}{3b+8}$

26) $v+3-\frac{5}{3v-9}$

27) $x-7-\frac{7}{4x-5}$

28) $n-7-\frac{3}{4n+5}$

29) $a^2+8a-7-\frac{6}{a+7}$

30) $8k^2-2k-4+\frac{5}{k-8}$

31) $x^2-4x-10-\frac{1}{x+4}$

32) x^2-8x+7

33) $3n^2-9n-10-\frac{8}{n+6}$

34) $k^2-3k-9-\frac{5}{k-1}$

35) $x^2-7x+3+\frac{1}{x+7}$

36) $n^2+9n-1+\frac{3}{2n+3}$

37) $p^2+4p-1+\frac{4}{9p+9}$

38) $m^2-8m+7-\frac{7}{8m+7}$

39) $r^2+3r-4-\frac{8}{r-4}$

40) $x^2+3x-7+\frac{5}{2x+6}$

41) $6n^2-3n-3+\frac{5}{2n+3}$

42) $6b^2+b+9+\frac{3}{4b-7}$

43) $v^2-6v+6+\frac{1}{4v+3}$

Answers - Chapter 6

6.1

Answers - Greatest Common Factor

1) $9+8b^2$

2) $x-5$

3) $5(9x^2-5)$

4) $1+2n^2$

5) $7(8-5p)$

6) $10(5x-8y)$

7) $7ab(1-5a)$

8) $9x^2y^2(3y^3-8x)$

9) $3a^2b(-1+2ab)$

10) $4x^3(2y^2+1)$

11) $-5x^2(1+x+3x^2)$

12) $8n^5(-4n^4+4n+5)$

13) $10(2x^4-3x+3)$

14) $3(7p^6+10p^2+9)$

15) $4(7m^4+10m^3+2)$

16) $2x(-5x^3+10x+6)$

17) $5(6b^9+ab-3a^2)$

18) $3y^2(9y^5+4x+3)$

19) $-8a^2b(6b+7a+7a^3)$

20) $5(6m^6+3mn^2-5)$

21) $5x^3y^2z(4x^5z+3x^2+7y)$

22) $3(p+4q-5q^2r^2)$

23) $10(5x^2y+y^2+7xz^2)$

24) $10y^4z^3(3x^5+5z^2-x)$

25) $5q(6pr-p+1)$

26) $7b(4+2b+5b^2+b^4)$

27) $3(-6n^5+n^3-7n+1)$

28) $3a^2(10a^6+2a^3+9a+7)$

29) $10x^{11}(-4-2x+5x^2-5x^3)$

30) $4x^2(-6x^4-x^2+3x+1)$

31) $4mn(-8n^7+m^5+3n^3+4)$

32) $2y^7(-5+3y^3-2xy^3-4xy)$

6.2

Answers - Grouping

1) $(8r^2-5)(5r-1)$

2) $(5x^2-8)(7x-2)$

3) $(n^2-3)(3n-2)$

4) $(2v^2-1)(7v+5)$

5) $(3b^2-7)(5b+7)$

6) $(6x^2+5)(x-8)$

7) $(3x^2+2)(x+5)$

8) $(7p^2+5)(4p+3)$

9) $(7x^2-4)(5x-4)$

10) $(7n^2-5)(n+3)$

11) $(7x+5)(y-7)$

12) $(7r^2+3)(6r-7)$

13) $(8x+3)(4y+5x)$

14) $(3a+b^2)(5b-2)$

15) $(8x+1)(2y-7)$

16) $(m+5)(3n-8)$

17) $(2x+7y^2)(y-4x)$

18) $(m-5)(5n+2)$

19) $(5x-y)(8y+7)$

20) $(8x-1)(y+7)$

21) $(4u+3)(8v-5)$

22) $2(u+3)(2v+7u)$

23) $(5x+6)(2y+5)$

24) $(4x-5y^2)(6y-5)$

25) $(3u-7)(v-2u)$

26) $(7a-2)(8b-7)$

27) $(2x+1)(8y-3x)$

6.3

Answers - Trinomials where a = 1

1) $(p+9)(p+8)$

2) $(x-8)(x+9)$

3) $(n-8)(n-1)$

4) $(x-5)(x+6)$

5) $(x+1)(x-10)$

6) $(x+5)(x+8)$

7) $(b+8)(b+4)$

8) $(b-10)(b-7)$

9) $(x-7)(x+10)$

10) $(x-3)(x+6)$

11) $(n-5)(n-3)$

12) $(a+3)(a-9)$

13) $(p+6)(p+9)$

14) $(p+10)(p-3)$

15) $(n-8)(n-7)$

16) $(m-5n)(m-10n)$

17) $(u-5v)(u-3v)$

18) $(m+5n)(m-8n)$

19) $(m+4n)(m-2n)$

20) $(x+8y)(x+2y)$

21) $(x-9y)(x-2y)$

22) $(u-7v)(u-2v)$

23) $(x-3y)(x+4y)$

24) $(x+5y)(x+9y)$

25) $(x+6y)(x-2y)$

26) $4(x+7)(x+6)$

27) $5(a+10)(a+2)$

28) $5(n-8)(n-1)$

29) $6(a-4)(a+8)$

30) $5(v-1)(v+5)$

31) $6(x+2y)(x+y)$

32) $5(m^2+6mn-18n^2)$

33) $6(x+9y)(x+7y)$

34) $6(m-9n)(m+3n)$

6.4

Answers - Trinomials where a $\neq$ 1

1) $(7x-6)(x-6)$

2) $(7n-2)(n-6)$

3) $(7b+1)(b+2)$

4) $(7v+4)(v-4)$

5) $(5a+7)(a-4)$

6) Prime

7) $(2x-1)(x-2)$

8) $(3r+2)(r-2)$

9) $(2x+5)(x+7)$

10) $(7x-6)(x+5)$

11) $(2b-3)(b+1)$

12) $(5k-6)(k-4)$

13) $(5k+3)(k+2)$

14) $(3r+7)(r+3)$

15) $(3x-5)(x-4)$

16) $(3u-2v)(u+5v)$

17) $(3x+2y)(x+5y)$

18) $(7x+5y)(x-y)$

19) $(5x-7y)(x+7y)$

20) $(5u-4v)(u+7v)$

21) $3(2x+1)(x-7)$

22) $2(5a+3)(a-6)$

23) $3(7k+6)(k-5)$

24) $3(7n-6)(n+3)$

25) $2(7x-2)(x-4)$

26) $(r+1)(4r-3)$

27) $(x+4)(6x+5)$

28) $(3p+7)(2p-1)$

29) $(k-4)(4k-1)$

30) $(r-1)(4r+7)$

31) $(x+2y)(4x+y)$

32) $2(2m^2+3mn+3n^2)$

33) $(m-3n)(4m+3n)$

34) $2(2x^2-3xy+15y^2)$

35) $(x+3y)(4x+y)$

36) $3(3u+4v)(2u-3v)$

37) $2(2x+7y)(3x+5y)$

38) $4(x+3y)(4x+3y)$

39) $4(x-2y)(6x-y)$

40) $2(3x+2y)(2x+7y)$

6.5

Answers - Factoring Special Products

1) $(r+4)(r-4)$

2) $(x+3)(x-3)$

3) $(v+5)(v-5)$

4) $(x+1)(x-1)$

5) $(p+2)(p-2)$

6) $(2v+1)(2v-1)$

7) $(3k+2)(3k-2)$

8) $(3a+1)(3a-1)$

9) $3(x+3)(x-3)$

10) $5(n+2)(n-2)$

11) $4(2x+3)(2x-3)$

12) $5(25x^2+9y^2)$

13) $2(3a+5b)(3a-5b)$

14) $4(m^2+16n^2)$

15) $(a-1)^2$

16) $(k+2)^2$

17) $(x+3)^2$

18) $(n-4)^2$

19) $(x-3)^2$

20) $(k-2)^2$

21) $(5p-1)^2$

22) $(x+1)^2$

23) $(5a+3b)^2$

24) $(x+4y)^2$

25) $(2a-5b)^2$

26) $2(3m-2n)^2$

27) $2(2x-3y)^2$

28) $5(2x+y)^2$

29) $(2-m)(4+2m+m^2)$

30) $(x+4)(x^2-4x+16)$

31) $(x-4)(x^2+4x+16)$

32) $(x+2)(x^2-2x+4)$

33) $(6-u)(36+6u+u^2)$

34) $(5x-6)(25x^2+30x+36)$

35) $(5a-4)(25a^2+20a+16)$

36) $(4x-3)(16x^2+12x+9)$

37) $(4x+3y)(16x^2-12xy+9y^2)$

38) $4(2m-3n)(4m^2+6mn+9n^2)$

39) $2(3x+5y)(9x^2-15xy+25y^2)$

40) $3(5m+6n)(25m^2-30mn+36n^2)$

41) $(a^2+9)(a+3)(a-3)$

42) $(x^2+16)(x+4)(x-4)$

43) $(4+z^2)(2+z)(2-z)$

44) $(n^2+1)(n+1)(n-1)$

45) $(x^2+y^2)(x+y)(x-y)$

46) $(4a^2+b^2)(2a+b)(2a-b)$

47) $(m^2+9b^2)(m+3b)(m-3b)$

48) $(9c^2+4d^2)(3c+2d)(3c-2d)$

6.6

Answers - Factoring Strategy

1) $3(2a+5y)(4z-3h)$

2) $(2x-5)(x-3)$

3) $(5u-4v)(u-v)$

4) $4(2x+3y)^2$

5) $2(-x+4y)(x^2+4xy+16y^2)$

6) $5(4u-x)(v-3u^2)$

7) $n(5n-3)(n+2)$

8) $x(2x+3y)(x+y)$

9) $2(3u-2)(9u^2+6u+4)$

10) $2(3-4x)(9+12x+16x^2)$

11) $n(n-1)$

12) $(5x+3)(x-5)$

13) $(x-3y)(x-y)$

14) $5(3u-5v)^2$

15) $(3x+5y)(3x-5y)$

16) $(x-3y)(x^2+3xy+9y^2)$

17) $(m+2n)(m-2n)$

18) $3(2a+n)(2b-3)$

19) $4(3b^2+2x)(3c-2d)$

20) $3m(m+2n)(m-4n)$

21) $2(4+3x)(16-12x+9x^2)$

22) $(4m+3n)(16m^2-12mn+9n^2)$

23) $2x(x+5y)(x-2y)$

24) $(3a+x^2)(c+5d^2)$

25) $n(n+2)(n+5)$

26) $(4m-n)(16m^2+4mn+n^2)$

27) $(3x-4)(9x^2+12x+16)$

28) $(4a+3b)(4a-3b)$

29) $x(5x+2)$

30) $2(x-2)(x-3)$

31) $3k(k-5)(k-4)$

32) $2(4x+3y)(4x-3y)$

33) $(m-4x)(n+3)$

34) $(2k+5)(k-2)$

35) $(4x-y)^2$

36) $v(v+1)$

37) $3(3m+4n)(3m-4n)$

38) $x^2(x+4)$

39) $3x(3x-5y)(x+4y)$

40) $3n^2(3n-1)$

41) $2(m-2n)(m+5n)$

42) $v^2(2u-5v)(u-3v)$

6.7

Answers - Solve by Factoring

1) $7,-2$

2) $-4,3$

3) $1,-4$

4) $-\frac{5}{2},7$

5) $-5,5$

6) $4,-8$

7) $2,-7$

8) $-5,6$

9) $-\frac{5}{7},-3$

10) $-\frac{7}{8},8$

11) $-\frac{1}{5},2$

12) $-\frac{1}{2},2$

13) $4,0$

14) $8,0$

15) $1,4$

16) $4,2$

17) $\frac{3}{7},-8$

18) $-\frac{1}{7},-8$

19) $\frac{4}{7},-3$

20) $\frac{1}{4},3$

21) $-4,-3$

22) $8,-4$

23) $8,-2$

24) $4,0$

25) $\frac{8}{3},-5$

26) $-\frac{1}{2},\frac{5}{3}$

27) $-\frac{3}{7},-3$

28) $-\frac{4}{3},-3$

29) $-4,1$

30) $2,-3$

31) $-7,7$

32) $-4,-6$

33) $-\frac{5}{2},-8$

34) $-\frac{6}{5},-7$

35) $\frac{4}{5}, -6$

36) $\frac{5}{3}, -2$